Chapter 4
Introduction to Valuation:
Time Value of Money
Lump Sum Calculations:
1. Present Value
2. Future Value

Present

$1.00

Put $1 in a bank that pays 10% interest rate compounded 1 time a year.
(cash flow to you is $1 because it comes out of your pocket).

Bank pays 10% interest
$1 \times (1 + 0.1) = $1.10

Future

1 (years)

$1.10

Original Investment plus interest paid (cash flow to you is $1 + $0.10 because it comes into your pocket).

The Present Value is $1.00

The Future Value is $1.10
Formula to Calculate Future Value

$$FV = PV \left(1 + \frac{i}{n}\right)^{n \times x}$$

- $FV$ = Future Value
- $PV$ = Present Value
- $i$ = Annual Interest Rate (also: APR)
- $n$ = number of compounding periods per year
- $x$ = # of years

Example 1: If you put $10,000 in bank at an annual rate of 6%, compounded monthly for 10 years, how much will you have at maturity?

Future Value = $FV = ?$

Present Value = $PV =$ how much you deposit to day = $10,000$

Annual Rate $i$ = 6% or 0.06

# compounding per year $n = 12$

years $x = 10$

$$FV = 10,000 \left(1 + \frac{0.06}{12}\right)^{10 \times 12}$$

$$= 10,000 \left(1 + .005\right)^{120}$$

$$= 10,000 \times 1.81939673403228$$

$$= 181,939.797$$
Excel FV Function

Future value = FV = FV
Present value = PV = PV
Period rate = \( \frac{i}{n} = \text{rate} \)
Total periods = \( x \times n = n_{\text{per}} \)

\[
= FV \left( \text{rate}, n_{\text{per}}, -PV \right)
\]

Skip PMT argument

\[
= FV \left( \frac{i}{n}, x \times n, -PV \right)
\]

\[
= FV \left( .005, 120, -10000 \right)
\]

\[
= \$ 18,193.97
\]
Interest is in dollars.
Interest Rate is in decimal or fractional or percentage terms (percentage of investment or loan).

Simple Interest:
Interest Earned on the original investment only (or paid on original loan).

Example 2:
FV = investment = $100 (cash out of pocket)
\( i \) = Annual Interest Rate = \( 0.10 \) or 10% 
\( n \) = compounding periods per year = 1 
\( x \) = years = 4 
FV = Future value of investment = ?

Simple interest = $100 \times 0.1 = $10.00

<table>
<thead>
<tr>
<th>Year</th>
<th>Interest</th>
<th>Amount in Bank</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>$100</td>
</tr>
<tr>
<td>1</td>
<td>$10</td>
<td>10 + $100 = $110</td>
</tr>
<tr>
<td>2</td>
<td>$10</td>
<td>10 + $110 = $120</td>
</tr>
<tr>
<td>3</td>
<td>$10</td>
<td>10 + $120 = $130</td>
</tr>
<tr>
<td>4</td>
<td>$10</td>
<td>10 + $130 = $140</td>
</tr>
</tbody>
</table>
Compound Interest:

Interest earned on both original investment and interest reinvested from prior periods.

Example 3:

\[ \begin{align*}
\text{PV} &= \$100 \\
\text{i} &= 0.10 \text{ or } 10\% \\
\text{n} &= 1 \\
X &= 4 \\
\text{FV} &= ?
\end{align*} \]

Investment

\begin{align*}
\text{Annual Interest Rate} \\
\text{\# compounding periods per year} \\
\text{years} \\
\text{Future Value}
\end{align*}

\[ \begin{align*}
\text{FV}_{\text{year 1}} &= 100 + 100 \times 0.1 = 100 + 10 = 110 \\
\downarrow \\
\text{FV}_{\text{year 2}} &= 110 + 110 \times 0.1 = 110 + 11 = 121 \\
\downarrow \\
\text{FV}_{\text{year 3}} &= 121 + 121 \times 0.1 = 121 + 12.1 = 133.10 \\
\downarrow \\
\text{FV}_{\text{year 4}} &= 133.10 + 133.10 \times 0.1 = 133.10 + 13.31 \\
&= 146.41
\end{align*} \]

Compound Interest FV \( \approx \) \$146.41

Simple Interest FV \( \approx \) \$140.80

Interest on Interest \( \approx \) \$6.41

Compounding:
The process of accumulating interest in an investment over time to earn more interest

Interest Earned on the reinvestment of previous interest payments
Math:

\[ P = 100 \]
\[ i = -0.10 \]
\[ n = 1 \]
\[ X = 1 \]
\[ FV = ? \]

Original investment interest

\[ FV = 100 + 100 \times 0.1 \]
\[ FV = 100 + 10 \]
\[ FV = 110 \]

Notice: 100 in both places

\[ 100 + 100 \times 0.1 \]

Plus sign

Notice:

\[ 100 \times 1 + 100 \times 0.1 \]

If we put 1 here, it is still the same

Notice that we can factor: (distributive property backwards)

\[ 100 \times 1 + 100 \times 0.1 \]
\[ 100 \times (1 + 0.1) = 100 \times 1.1 \]

Conclusion: 100 + 100 \times 0.1 = 100 \times 1.1
Derive Easier Formula for Compound Interest

Lump sum future value calculation:

Present value \( PV = 100 \)

Annual rate \( i = 0.10 \)

Number of comp. periods per year \( n = 1 \)

Year future value \( FV = ? \)

\[ FV_1 = PV \times (1 + i)^n \]

\[ FV_1 = 100 \times (1 + 0.1)^1 = 110 = 100(1 + 0.1) \]

\[ FV_2 = FV_1 \times (1 + i) \]

\[ FV_2 = 110 + 110 \times 0.1 = 121 \]

\[ FV_2 = 100(1 + 0.1) + 100(1 + 0.1)^2 \times 0.1 \]

\[ FV_3 = FV_2 \times (1 + i)^2 \]

\[ FV_3 = 121 + 121 \times 0.1 = 133.1 \]

\[ FV_3 = 100(1 + 0.1)^2 + 100(1 + 0.1)^2 \times 0.1 \]

\[ FV_4 = FV_3 \times (1 + i)^3 \]

\[ FV_4 = 133.1 + 133.1 \times 0.1 = 146.41 \]

\[ FV_4 = 100(1 + 0.1)^3 + 100(1 + 0.1)^3 \times 0.1 \]

\[ FV_4 = 100(1 + 0.1)^4 = 146.41 \]
Fundamental Truth in Finance

A dollar received today is worth more than a dollar received later. (This is true because of interest).

\[ 0 \quad 1 \quad 2 \quad \text{years} \]

- **Current Value**
  - $1.00
  - Dollar received today

- **Future Value**
  - $1.00
  - Dollar received in one year

Which would you prefer?

Future value $1.00 = 1.10$

If you took the $1 today and earned interest:

- **Present Value**
  - $1.00

- **Future Value**
  - $1.10
  - The $1 you received the first year.

After you invest $1.00 at 10% interest for one year, you can earn $0.10 more.

The $1 you received that first year, is $0.10 more than you'd have if you didn't invest.

Yes! This is the Fundamental Truth in Finance.
$1.00
present value

Bank pays 10% Interest
$1.00 \times (1.1) = $1.10

$1.00
Present value

$0.10
Interest

$1.10
Future value

The future value of $1.00 is $1.10

$1.00
The present value of $1.10 is $1.00

This way you add interest

This way you take interest out (subtract out)
\[ FV = PV \left(1 + \frac{i}{n}\right)^{n \times n} \]

Divide both sides:
\[
\frac{FV}{\left(1 + \frac{i}{n}\right)^{n \times n}} = \frac{PV \left(1 + \frac{i}{n}\right)^{n \times n}}{\left(1 + \frac{i}{n}\right)^{n \times n}}
\]

Cancel:
\[
\frac{FV}{\left(1 + \frac{i}{n}\right)^{n \times n}} = \frac{PV \left(1 + \frac{i}{n}\right)^{n \times n}}{\left(1 + \frac{i}{n}\right)^{n \times n}}
\]

PV on one side:
\[
\frac{FV}{\left(1 + \frac{i}{n}\right)^{n \times n}} = PV
\]

Formula:
\[
PV = \frac{FV}{\left(1 + \frac{i}{n}\right)^{n \times n}}
\]

Derive formula for PV
Formula to calculate Present Value

\[ PV = \frac{FV}{(1 + \frac{i}{n})^{x\times n}} \]

FV = Future Value
PV = Present Value
\( i \) = Annual Interest Rate = Discount Rate
n = # of compounding periods per year
x = # of years
\( \frac{i}{n} \) = period Rate
\( x\times n \) = Total number of periods.

Excel PV Function

Future Value = \( FV = FV \)
Present Value = \( PV = PV \)
Period rate = \( \frac{i}{n} = rate \)
Total periods = \( x\times n = n_{per} \)

\[ PV = PV \left( \frac{i}{n}, x\times n, FV \right) \quad \text{skip PMT by putting 2 commas} \]

\[ PV = PV \left( \frac{i}{n}, x\times n, , FV \right) \]
Example 6: How much do we have to put in the bank today \((i = 0.1, n = 12)\) to be a millionaire in 40 years?

\[
FV = 1,000,000 \\
i = 0.1 \\
n = 12 \\
x = 40 \\
\]

\[
PV = \frac{1,000,000}{(1 + \frac{0.1}{12})^{12 \times 40}} = 18,621.74
\]

\[
= PV\left(\frac{0.1}{12}, 12 \times 40, 1,000,000\right) = -18,621.74
\]

Notice that Excel knows that the money going into the investment has a negative cash flow.

**Answer:**

If we want to be a millionaire in 40 years and we could earn 10% compounded monthly, we would need to invest $18,621.74 today.

\[
\frac{1,000,000}{18,621.74} = 981,378.26
\]

Interest earned

---

Time

\[0 \rightarrow 1 \\
1 \rightarrow 2 \\
2 \rightarrow 39 \\
39 \rightarrow 40 \\
you know what you want in FV

-18,621.74

today

981,378.26

Interest going backwards "taking out all interest to see what to deposit"
Example 7: How much would we have to invest today, if we wanted to have $150,000 for our daughter's college tuition in 18 years and we could earn an annual interest rate (discount rate) of 6.95% compounded daily (365 times a year)?

\[ PV = ? \]
\[ FV = \$150,000 \]
\[ n = 18 \]
\[ i = 0.0695 \Rightarrow 6.95\% \]
\[ N = 365 \]

\[ PV = \frac{FV}{(1 + \frac{i}{n})^{n\times x}} \]
\[ PV = \frac{150,000}{(1 + \frac{0.0695}{365})^{365\times 18}} = \frac{150,000}{(1.0001904109580411)^{6570}} = \$42,937.88 \]

Excel:
\[ PV = \frac{150,000}{(1 + \frac{0.0695}{365})^{365\times 18}} = \$42,937.88 \]

Answer: If we want to have $150,000 in 18 years to pay for our daughter's college tuition & we can earn 6.95% compounded daily, we would need to invest $42,937.88 today.

Notice: Excel shows this as negative because this is a cash flow out of your wallet & into your bank.
Example 8:

If you want to buy a $350,000 C & C Router Machine to improve manufacturing efficiency and you have $200,000 today that you can invest at an annual rate of 8.5% compounded monthly, how long do you have to wait until your investment will grow to $350,000. (Assume machine will cost 350,000 in future).

\[ PV = 200,000 \]
\[ i = 0.085 \]
\[ n = 12 \]
\[ x = ? \]
\[ x \times n = ? \]
\[ FV = 350,000 \]

**MATH**

\[ FV = PV \times \left(1 + \frac{i}{n}\right)^{nx} \]

\[ 350,000 = 200,000 \times \left(1 + \frac{0.085}{12}\right)^{12x} \]

\[ \frac{35}{20} = \left(1.007083\right)^{nx} \]

\[ \frac{\ln\left(\frac{35}{20}\right)}{\ln\left(1.007083\right)} = nx \]

\[ 79.2841 = nx = \text{total periods} \]

\[ \frac{79.2841}{12} = x = \text{years} \]

6.607 = years

**Answer:** If you have $200,000 to invest today at 8.5% compounded monthly and you need $350,000 to buy the machine, you would have to wait 6.607 years.
Example 9:
If you want to buy a $350,000 C & C Router to improve manufacturing efficiency and you can invest $250,000 today for the next five years (compounding 2 times a year), what annual interest rate (APR) do you need to find so that you can afford the machine?

\[ PV = 250,000 \]
\[ n = 2 \]
\[ x = 5 \]
\[ FV = 350,000 \]
\[ \frac{i}{n} = \text{period rate} \]

Math:

\[ FV = PV \times \left(1 + \frac{i}{n}\right)^{nx} \]
\[ 350,000 = 250,000 \times \left(1 + \frac{i}{2}\right)^{2 \times 5} \]
\[ \frac{35}{25} = \left(1 + \frac{i}{2}\right)^{10} \]
\[ \left(\frac{7}{5}\right)^{\frac{1}{10}} = 1 + \frac{i}{2} \]
\[ 1.034219694 = 1 + \frac{i}{2} \]
\[ 1.034219694 - 1 = \frac{i}{2} \]
\[ 0.034219694 = \frac{i}{2} \]
\[ 0.034219694 \times 2 = i \]
\[ 0.068439388 = i \]

Excel: solve for rate

\[ i = \text{Rate}(nper, ; PV, FV) = \{\text{Period}\}; i \]
\[ i = \text{Rate}(n \times x, ; PV, FV) \]
\[ i = \text{Rate}(10, ; 250,000, 350,000) \]
\[ .034219694 = \frac{i}{n} = \text{half year rate} \]
\[ i = .034219694 \times 2 = .068439388 \]

Answer:
If you want to buy the $350,000 machine & you have $250,000 to invest today for the next 5 years (compounded semi annually), you would need an APR of 6.84%.
The formula for the "Time Value of Money" (TVM) forms the basis for many of the most common financial transactions we take for granted each day. TVM explains why a dollar today is worth more than a dollar in the future. TVM is why most of us don't keep all of our money under our mattress. We understand that inflation will slowly erode the value of the savings over time and it's a better idea to invest the money in some vehicle with the potential to generate interest like a stock or savings account.

While the evidence is not definitive, financial archeologists believe that Leonardo of Pisa, commonly known as "Fibonacci", was the first person to publish the formulas that have evolved into what we now know as TVM. It should be noted that much Fibonacci's work was built on top of prior work by Arabic and Indian mathematicians.

Fibonacci published his work in a book called Liber Abaci in 1202. Given the period in which he lived, its no surprise that Fibonacci's work focused on everyday mercantile issues like the TVM, interest rates and foreign exchange rates. Fibonacci published his work at a time when Italy was a collection of independent city states, each with their own currency trading with other merchants from Northern Africa, China, the Middle East and other parts of the Mediterranean.

Liber Abaci also included groundbreaking research into another area of mathematics that should be familiar to today's technical trader. The Fibonacci Series is a sequence of numbers where the each number is the sum of the prior two numbers in the series (1, 1, 2, 3, 5, 8...). While the series itself may not seem that exciting, the quotient of sequential numbers is always 1.618 or the inverse 0.618 which is more commonly referred to as the "Golden Mean". The Golden Mean is found throughout nature and human endeavors. In nature the number of female bees in a hive divided by male bees is always close to 1.618 and the Golden Mean can also be found in the spirals of a nautilus shell. The Golden Mean is also prevalent in modern and ancient architecture. Perhaps the most famous example is the Parthenon in Greece whose proportions adhere to the Golden Mean.

The Fibonacci Series also has a place in modern trading. Many investor use Fibonacci numbers in technical analysis to generate lines of support and resistance in stocks. According to Investopedia, the most popular Fibonacci studies are arcs, fans, retracements and time zones.

http://post.polls.yahoo.com/quiz/financeresults.php
Rule of 72 for estimating the rate you will need to have your investment double.

Rule of 72: \[ \frac{n}{r} \approx \frac{72}{n \times x} \]

If you want to double your investment in 5 years where the interest is compounded monthly.

\[ \frac{n}{r} \approx \frac{72}{n \times 5} = \frac{72}{12 \times 5} = \frac{72}{60} = 1.2 \]

\[ r \approx 1.2 \times 12 = 14.4 \]

But this is not .144

So to get final estimate:

\[ \frac{14.4}{100} = 0.144 \approx r \]