Annuities

1) What is an Annuity?
    1. Two possibilities for an annuity
        i. Pay an equal amount of money each period
            1. Example~~: Home Mortgage payment or Car loan payment~~ Retirement Plan
                i. If you can save $200 at the end of each month for the next 35 years and you can earn 12% compounded monthly, how much will you have when you retire?
                    1. This is an example of a "Future Value of an Annuity" calculation where we solve for the Future Value
            2. Example: Retirement Plan
                i. If you need want to be a millionaire in 35 years and you can earn 12% compounded monthly, how much do you need to deposit at the end of each month?
                    1. This is an example of a "Future Value of an Annuity" calculation where we solve for the Payment
            3. Example: Savings Plan
                i. If a business needs to $100,000 in 10 years and you can earn 10% compounded quarterly, how much do you need to deposit at the end of each ~~month?~~ quarter?
                    1. This is an example of a "Future Value of an Annuity" calculation where we solve for the Payment. This is also an example of a "Sinking Fund".
            4. Example: How much can we afford for the new car
                i. Example: If you can afford to pay $250 at the end of each month for the next 5 years at 6% compounded monthly, how much do you have today to spend on an automobile purchase?
                    1. This is an example of a "Present Value of an Annuity" calculation where we solve for the Present Value

        ii. Receive an equal amount of cash each period
            1. Example: Retirement plan
                i. If you have $500,000 when you retire and you plan to live for 40 years, and you can earn 8% compounded monthly, how much can you withdraw each month from your account?
                    1. This is an example of a "Present Value of an Annuity" calculation where we solve for the Payment

2) Define an annuity:
    1. All cash flow payments are equal in amount
    2. The time between each payment is equal

3) Two types of annuities:
    1. Ordinary Annuities (all annuities are Ordinary unless otherwise stated).
        i. Period payment is made at the end of each period
        ii. Example: Home Mortgage payment or Car loan payment
    2. Annuity Due
        i. Period payment is made at the beginning of each period
        ii. Example: Lease payments

Future Value Annuities: Chapter 10.1

1) Future Value Annuities
    1. Instead of depositing one lump sum, waiting for compound interest to increase the value, and then withdrawing the FV amount, we will make a series of equal deposits or payments made at regular time intervals, wait for compound interest to increase the value, and then withdrawing the FV amount.
        - Each deposit or "payment" is for the same amount.
        - The time between each payment is exactly the same and is called "payment period."
        - Each payment is made at the end of the payment period.
        - When the FV annuity starts, no payment is made.
        - "Term of the annuity" means the time from the beginning of the first payment period to the end of the last payment period.
        - Future Value of an annuity is the final value of all the compounded payments

2) Formulas:

Variables for the Financial Functions Defined:

An = Annuity = Regular Payments (PMT) Made at Regular Time Intervals

Made at End of Period

LS = Lump Sum Payment = Payment Made Once

FV=Future Value (Lump Sum Value in the Future)

PV=Present Value (Lump Sum Value in the Present)

PMT = Regular Payment Made at Regular Time Intervals

i = Annual Interest Rate

n = Number of Compounding Periods per Year

x = Years

FV of regular payments at regular intervals (Retirement Plan: find out how much we will have when we retire?)

$$FV_{An} = PMT*\frac{\left(1+\frac{i}{n}\right)^{n*x} - 1}{\left(\frac{i}{n}\right)}$$

Excel:
use FV function

PMT for FV of regular payments at regular intervals (Retirement Plan: How much should we deposit each period?)

$$PMT_{FV_{An}} = \frac{FV_{An}}{\left(\frac{\left(1+\frac{i}{n}\right)^{n*x} - 1}{\left(\frac{i}{n}\right)}\right)}$$

Excel:
use PMT function

Example 1: If you can save $200 at the end of each month for the next 35 years and you can earn 12% compounded monthly, how much will you have when you retire?

[redacted]

$PMT = 200$

$X = 35$

$i = .12$

$n = (\text{Monthly}) = 12$

$FV = ?$

$\frac{i}{n} = \frac{.12}{12} =$ period rate

$n * X = 12 * 35 =$ total periods

$$FV = PMT * \left[\frac{\left(1 + \frac{i}{n}\right)^{\wedge(n*x)} - 1}{\frac{i}{n}} \right]$$

$$FV = 200 * \left[\frac{\left(1 + \frac{.12}{12}\right)^{\wedge(12*35)} - 1}{\frac{.12}{12}} \right]$$

$$= 200 * \left[\frac{1.01^{\wedge 420} - 1}{.01} \right]$$

$$= 200 * \frac{65.3095947145687}{.01}$$

$$= 200 * 6530.95947145687$$

$$= 1286191.8942914$$

$$\approx \$1,286,191.89$$

If you can save $200 per month for the next 35 years and earn 12% companded monthly, you will have $1,286,191.89 when you retire

Example 2: If you want to be a millionaire in 35 years and you can earn 12% compounded monthly, how much do you need to deposit at the end of each month?

$FV = \$1,000,000$ $\qquad \frac{i}{n} = \frac{.12}{12} =$ period rate

$X = 35$ years $\qquad n * x = 12 * 35 = \left\{ \begin{matrix} \text{total \#} \\ \text{of periods} \end{matrix} \right\}$

$i = 12\% \Rightarrow .12$

$n =$ (Monthly) $= 12$

$PMT = ??$

$$PMT = \frac{FV}{\left[\dfrac{\left(1+\frac{i}{n}\right)^{\wedge(n*x)} - 1}{\frac{i}{n}} \right]}$$

$$PMT = \frac{1,000,000}{\left[\dfrac{\left(1+\frac{.12}{12}\right)^{\wedge(12*35)} - 1}{\frac{.12}{12}} \right]}$$

$$PMT = \frac{1,000,000}{6430.9594714687}$$

$$= 155.4977922\,7165$$

$$\approx \$155.50$$

If you want to be a millionaire in 35 years & you can earn 12% compounded monthly, you would need to deposit $155.50 at the end of each month

Example 3: If you can save $200 at the beginning of each month for the next 35 years and you can earn 12% compounded monthly, how much will you have when you retire?

This says the beginning of each month. This means that this is an "Annuity Due"!

Rules for Annuity Due

1. Add 1 to the total number of periods.
2. calculate your answer.
3. subtract 1 PMT amount from the Future Value.

The FV formula for Annuity Due

$$FV = PMT * \left[\frac{(1+\frac{i}{n})^{\wedge(n*x+1)} - 1}{\frac{i}{n}} \right] - PMT$$

variables

FV = ?
PMT = 200
x = 35
i = 12%
n = 12
$\frac{i}{n} = \frac{.12}{12} = .01$
$n*x+1 = 12*35+1 = 421$

$$FV = 200 * \left[\frac{(1+.01)^{\wedge 421} - 1}{.01} \right] - 200$$

$$= 200 * \left[\frac{65.96269066 - 1}{.01} \right] - 200$$

$$= 200 * \left[\frac{64.96269066}{.01} \right] - 200$$

$$= 200 * 6496.269066 - 200$$

$$= 1299253.8132343 - 200$$

$$= 1299053.81|32343$$

$$= \$1,299,053.81$$

Given the above details our future value would be $1,299,053.81

Example 4: If your employer will match your retirement savings plan monthly contributions of $225, and you can earn 9% compounded [redacted] for the next 25 years, how much will you have when you retire?

Monthy

FV = ??

→ you put in = $225

→ employer matches * 2

PMT = 450

$\frac{i}{n} = \frac{.09}{12}$

$n * x = 12 * 25 = 300$

$i = .09$

$n = 12$

$x = 25$

$$FV = PMT * \left[\frac{\left(1+\frac{i}{n}\right)^{\wedge(n*x)} - 1}{\frac{i}{n}}\right]$$

$$FV = 450 * \left[\frac{\left(1+\frac{.09}{12}\right)^{\wedge(12*25)} - 1}{\frac{.09}{12}}\right]$$

$FV = 450 * 1121.1219373178$5

$= 504504.87|1793$

$\approx$ $504,504.87

Total you contributed = 225 * 300 = $135,000

total you got = 504,504.87

total gain = 504,504.87 − 135,000 = $369,504.87

total received = 504504.87

total gain = 369504.87

IRA = "Individual Retirement Account"

- IRA = "Individual Retirement Account"
- Tax deferred Retirement FV annuity plan
- Reduces current income taxes paid
- Builds savings for retirement
- Each year you are allowed to deposit approximately $4000 into your IRA
- The $4000 amount is deductible on your taxes
- You do not have to pay income taxes until you withdraw it after retirement
- At that time it is presumed that you will have a lower income and a lower tax percentage to pay.

Present Value Annuities: Chapter 10.2

1) Present Value Annuities
    1. The lump sum that you could deposit today so that equal periodic withdrawals could be made.
        - Each withdrawal or "payment" is equal in amount.
        - The time between each payment is exactly the same and is called "payment period."
        - Each payment is made at the end of the payment period.
        - "Term of the annuity" means the time from the beginning of the first withdrawal period to the end of the last withdrawal period.
        - Present Value of an annuity is the start value of all the compounded payments.
        - Payments are made at the end of each period, and yet, the PV of the annuity is valued at the beginning of the first period

    2. Formulas:

Variables for the Financial Functions Defined:

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Made at End of Period

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FV=Future Value (Lump Sum Value in the Future)

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PMT = Regular Payment Made at Regular Time Intervals

i = Annual Interest Rate

n = Number of Compounding Periods per Year

x = Years

PV of regular payments at regular intervals (Retirement Plan: What lump sum deposited today do we need so that you can take out future PMT?)

$$PV_{An} = PMT*\frac{1 - \left(1+\frac{i}{n}\right)^{-(n*x)}}{\left(\frac{i}{n}\right)}$$

In Excel:

use PV function

PMT for PV of regular payments at regular intervals (Retirement Plan: How much can we withdraw each period after we retire?)

$$PMT_{PV_{An}} = \frac{PV_{An}}{\left(\frac{1 - \left(1+\frac{i}{n}\right)^{-(n*x)}}{\left(\frac{i}{n}\right)}\right)}$$

In Excel:

use PMT function

Example 1: If you can afford to pay \$250 at the end of each month for the next 5 years at 6% compounded monthly, how much do you have today to spend on an automobile purchase?

Loan Question!

Auto Price today = PV = ?

PMT = 250

$x = 5$

$n = 12$

$i = 6\% = .06$

$\frac{i}{n} = \frac{.06}{12} = .005$

$n \times x = 12 \times 5 = 60$

$$PV = PMT \times \left[\frac{1 - \left(1 + \frac{i}{n}\right)^{\wedge(-n \times x)}}{\frac{i}{n}} \right]$$

$$PV = PMT \times \left[\frac{1 - \left(1 + \frac{.06}{12}\right)^{\wedge(-12 \times 5)}}{\frac{.06}{12}} \right]$$

$$PV = 250 \times \left[\frac{1 - (1 + .005)^{\wedge(-60)}}{.005} \right]$$

$$PV = 250 \times \left[\frac{1 - (1.005)^{\wedge -60}}{.005} \right]$$

$$PV = 250 \times \left[\frac{1 - .258627804}{.005} \right]$$

$$PV = 250 \times \frac{.741372196}{.005}$$

$$PV = 250 \times 51.72556075$$

$$PV = 12931.3901877826$$

PV = \$12,931.39

If you can afford to make a PMT at the end of each month for the next 5 years & you can find a loan for 6% compounded monthy monthy, then you can afford a \$12,931.39 price tag.

Example 2: If you have \$500,000 when you retire and you plan to live for 40 years, and you can earn 8% compounded monthly, how much can you withdraw each month from your account?

The key to this problem is to see that you are standing at the retirement date looking into the future. This makes the \$500,000 the PV and PMTs for the next 40 years in the future.

$PV = \$500,000$ $\qquad \frac{i}{n} = \frac{.08}{12}$

$x = 40$ $\qquad n * x = 12 * 40$

$i = 8\%$

$n = 12$

$$PMT = \frac{PV}{\left[\frac{1 - \left(1 + \frac{i}{n}\right)^{\wedge -(n * x)}}{\frac{i}{n}}\right]}$$

$$PMT = \frac{500,000}{\left[\frac{1 - \left(1 + \frac{.08}{12}\right)^{\wedge(-12 * 40)}}{\frac{.08}{12}}\right]}$$

$$PMT = \frac{500,000}{143.82039230837}$$

$$PMT = 3476.558449$$

$$PMT \approx \$3476.55$$

on the day we retire, if we have \$500,000 & we plan to live for 40 years & we can earn 8% compounded monthly, we can withdraw \$3476.55 at end of each month.

Example 3: If you are offered $200,000 in your hands today or $80,000 in your hands today and payments of $10,000 at the end of each quarter for the next 4 years (assume that money could be invested at 8% compounded quarterly), which would you choose?

see excel sheet tab PV(3)

Future Value Annuities (Sinking Funds): Chapter 10.3

Sinking Fund:

- Periodic payments are made into an account that will result in a desired FV amount.
- The FV amount is known
- The number of periods is known
- The interest rate per period is known
- We want to find the equal periodic payment!

Examples:

- Want to buy a building in the future, but you don't know how much to deposit each period
- Some companies are required to set up sinking funds when then have to pay back a loan in the future.
- Government and corporations set up sinking funds to pay off bonds that are due in the future.

Example 1:
Pete's Coffee Inc. needs to pay off a $100,000 loan in 10 years. If the interest rate for a sinking fund is 8% compounded annually, what is the periodic payment Pete's must make at the end of each year?

$FV = 100{,}000$

$X = 10$

$i = 8\% \Rightarrow .08$

$n = 1$

$PMT = ?$

$$PMT = \frac{FV}{\left[\frac{\left(1+\frac{i}{n}\right)^{\wedge(n*X)} - 1}{\frac{i}{n}}\right]}$$

$\boxed{\text{Savings Plan}}$

$$PMT = \frac{100{,}000}{\left[\frac{\left(1+\frac{.08}{1}\right)^{\wedge(1*8)} - 1}{.08}\right]} = \$6{,}902.95$$

(Rounded)

If Peets' wanted to pay off the $100,000 in 10 years, & they could earn 8% compounded annually, the periodic PMT would be $6902.95

Example 2:
In four years, Ajax Coal Company plans to purchase a new D11N Caterpillar tractor for their open pit coal mine. It currently sells for $758,000, but the price in increasing at 6% per year compounded semiannually. Ajax decides to set up a sinking fund through Carla Fresquez at Merrill Lynch, in order to save up to buy the tractor. Find the amount of each payment into the sinking fund, if annual payments are made and the money is expected to earn 8% compounded annually.

step 1

Lump sum!

$x = 4$

$PV = 758{,}000$

$i = .06$

$n = 2$

$$FV = PV * \left(1 + \frac{i}{n}\right)^{\wedge(n*x)}$$

Find FV for Price of Tractor

$$758000 * \left(1 + \frac{.06}{2}\right)^{\wedge(4*2)} = \$960{,}211.72$$

step 2 find amount we need to deposit each period into the sinking fund in order to buy the tractor in 4 years.

$FV = 960{,}211.72$

$x = 4$

$n = 1$

$i = .08$

$$PMT = \frac{FV}{\left[\frac{\left(1+\frac{i}{n}\right)^{\wedge(n*x)} - 1}{i/n}\right]} = \frac{960{,}211.72}{\left[\frac{\left(1+\frac{.08}{1}\right)^{\wedge 4} - 1}{\frac{.08}{1}}\right]}$$

$PMT = \$213090.96$

Taking into consideration of price increases for the tractor, the PMT for our sinking fund must be $ 213,090.96

Example 3:
Sinking Fund Table

see table in Excel sheet
"sink(3)"