Basics of Probability

*spelling errors may occur because No red squiggly line came up.

1 Probability

a. probability = chance = likelihood

b. probability = chance that something will occur in future

or

A Numerical measure of the likelihood that an event will occur

If E_i = i^{th} experimental outcome (sample point)

$P(E_i)$ = probability of Event i (like roll a 6)

$P(6) = \frac{1}{6}$

c. $0 \leq P(E_i) \geq 1$ for all i

d. $P(E_i)$ is never know with certainty (only estimate)

e. $P(E_i)$ is an estimate of an event that <u>may</u> occur in <u>future</u>

2 Experiment

a. A process that generates well-defined outcomes. On any single repetition of an experiment, one and only one of the possible experimental outcomes can occur.

or

Any Activity that has 2 or more possible results and it is uncertain which will occur.

"outcomes" → results

b. Examples:

Experiment		Experimental outcomes (sample points)
1. Toss a coin	=>	Head, Tail
2. Drive on Bridge	=>	stuck in Traffic, Not stuck
3. Roll Die	=>	1, 2, 3, 4, 5, 6
4. Make product	=>	Defect, Not Defect

3 Sample point (Experimental outcome)

ⓐ one of the experimental outcomes

ⓑ Example:

Experiment: Flip coin 1 time

sample Point = Head

sample Point = Tail

ⓒ Element of sample space

4 Sample space

ⓐ List of all possible experimental outcomes (sample points)

ⓑ Example:

{Sample space for Flip coin 1 time} = S = {Head, Tail}

{sample space for Flip coin 2 times} = S = {(H,H), (H,T), (T,T), (T,H)}

(each is sample point or 1 experimental outcome)

5 Multi-Step Experiment

ⓐ Experiment with more than 1 step.

ⓑ Example:

① Not a Multi-step Experiment: Flip coin 1 time

② Multi-step Experiments: - Flip Coin 3 times

- Drive across bridge 7 times

6 Tree Diagram

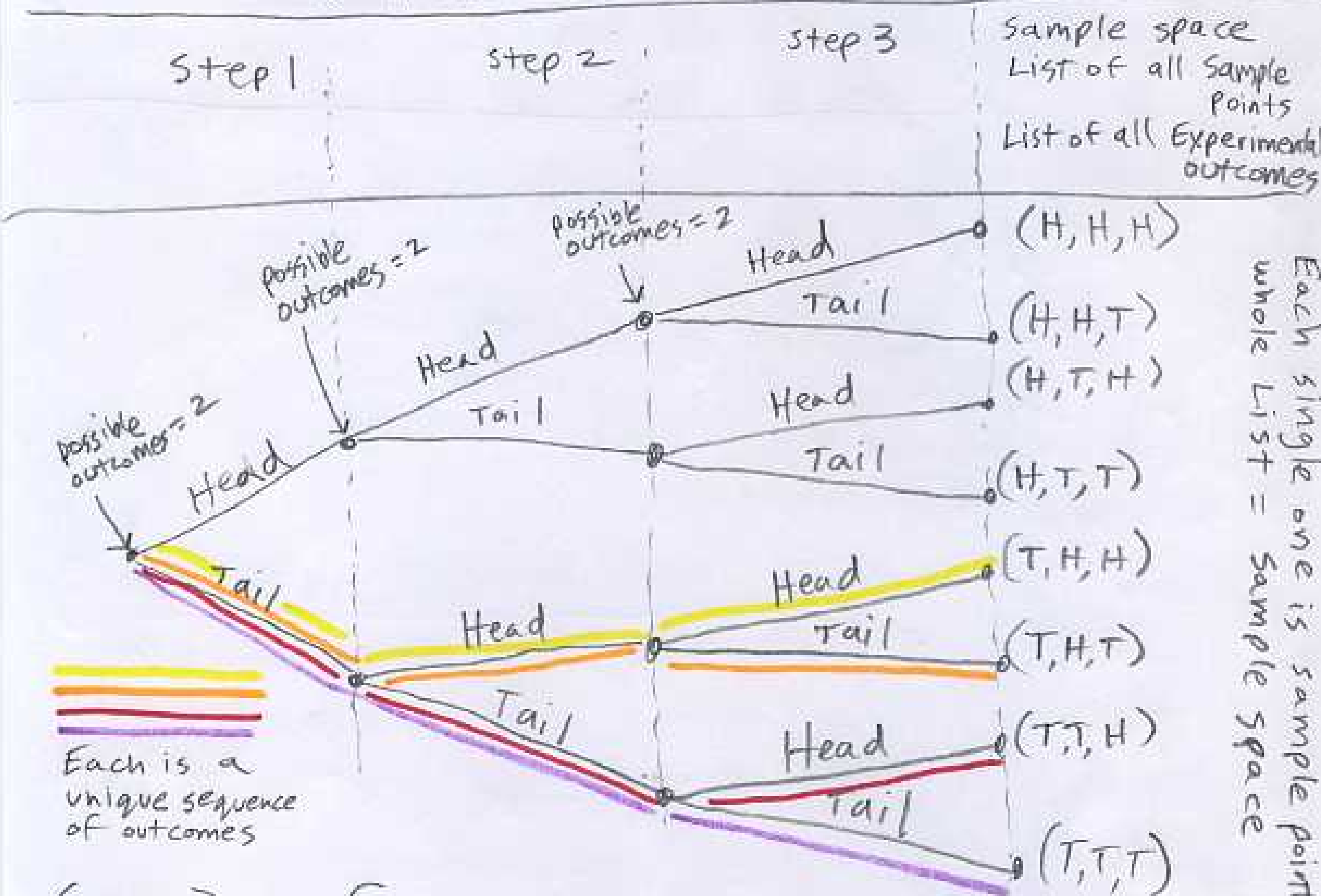
a) Graphical representation that helps to visualize mult-step experiments

b) Without Tree Diagrams, it is hard to count all experimental outcomes (sample points)

Tend to under count

c) Example:

Multi-step Experiment: Toss coin 3 times - to see if we can see any patterns for getting a Head or Tail



$$\begin{Bmatrix}\text{Sample}\\\text{Space}\end{Bmatrix} = S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

How many total outcomes in Experiment? Count = 8

(d) Alternative for visualizing Multi-step Experiment:

(Not as easy)

Experiment = Flip coin 3 times
H = Head T = Tail

	1st Toss	2nd Toss	3rd Toss	# Heads	Probability of Ex. outcome sample point
1	H	H	H	3	1/8 = .125
2	H	H	T	2	1/8 = .125
3	H	T	H	2	1/8 = .125
4	T	H	H	2	1/8 = .125
5	T	T	H	1	1/8 = .125
6	T	H	T	1	1/8 = .125
7	H	T	T	1	1/8 = .125
8	T	T	T	0	1/8 = .125

A Probabilities are: $0 \le P(E_i) \le 1$

$\left\{ \text{Add All Experimental outcome probabilities} \right\} = 1$

Each Probability is a number between 0 & 1

or

# of Heads in 3 Tosses	$P(E_i)$	$P(E_i)$
0	P(0 Heads)	1/8 = .125
1	P(1 Heads)	3/8 = .375
2	P(2 Heads)	3/8 = .375
3	P(3 Heads	1/8 = .125

all are between 0 & 1

$\sum P(E_i) = 1$

7 Counting Rule for Multi-step Experiment

a) {total Number of Experimental outcomes (sample points)} $n_1 * n_2 * \ldots * n_k$ = {size of sample space}

If n is same each step {total #} $= n^k$

K = number of steps in experiment

n_1 = number of possible outcomes step 1

n_2 = number of possible outcomes step 2

b) Example:

Experiment: Flip coin 3 times

$n_1 = 2$ (Heads or Tails)

$n_2 = 2$ (Heads or Tails)

$n_3 = 2$ (Heads or Tails)

$K = 3$

{Total number of sample points (Experimental outcomes)} $= 2 * 2 * 2 = 8$

Because n is same each time:

{total # outcomes} $= n^k = 2^3 = 8$

7) ⓒ Example:

Experiment: Roll 2 dice

$n_1 = 6$
$n_2 = 6$
$K = 2$

{Total # of sample points} $6 * 6 = 6^2 = 36$

sample space (List of sample points)

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

↳ sometimes you want to find total # of sample points (Experimental outcomes) in a slightly different situation.

$n! = n$ factorial

example: 1 $5! = 5*4*3*2*1$

$5! = 120$

2

$$\frac{10!}{7!} = \frac{10*9*8*\cancel{7}*\cancel{6}*\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}{\cancel{7}*\cancel{6}*\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}$$

$= 10*9*8$

$= 90*8$

$= 720$

DON'T FORGET:

$0! = 1$

example $\frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4!}{1} = 24$

Use Excel

=FACT(10)/FACT(7)

= 720

Remember:

$$\bar{x} = \frac{\Sigma x}{n}$$

$$\mu = \frac{\Sigma x}{N}$$

N = # of items in population
n = # of items in sample

we would like to be able to do this:

Find all possible combinations of sample size n, from a population with size N.

8 Count total number of experimental outcomes (sample points) when selecting n objects from a set of N objects (order does not matter $(1,2)=(2,1)$)

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a) Counting Rule for Combinations (order does not matter)

$$_{N}C_{n} = C_{n}^{N} = \binom{N}{n} = \frac{N!}{n!\,(N-n)!}$$

N = count of all objects (pop. size)
n = objects taken
items selected (sample size)
subset

9 Counting Rule for Permutations $\left(\begin{array}{c}\text{order matters} \\ (2,1) \neq (1,2)\end{array}\right)$

$$ {}_N P_n = P^N_n = \binom{N}{n} = \frac{N!}{(N-n)!} $$

p. 9

$${}_N C_n = \frac{N!}{n!\,(N-n)!}$$

Example ① che Dawson's Basketball Team has 12 players. How many 5 person teams can there be?

$n = 12$
$r = 5$

$${}_{12}C_5 = \frac{12!}{5!\,(12-5)!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot \cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{5\cdot 4\cdot 3\cdot 2\cdot 1 * \cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$$

$$= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{\cancel{12}\cdot 11\cdot \cancel{10}\cdot 9\cdot 8}{\cancel{12}\cdot \cancel{10}\cdot 1} = 11 * 9 * 8$$

$$= 99 * 8 = 792$$

$\boxed{\text{use Excel}}$

= COMBIN(12,5) = 792

$$_nP_r = \frac{N!}{(N-n)!}$$

Example ① Che wants to find out how many teams he can create from 12 players with each team having 5 players & he wants to rank each team (J, S, T, C, Z is different than S, J, T, C, Z)

$$_{12}P_5 = \frac{12!}{(12-5)!} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$$

$$= 95040$$

Use Excel

= PERMUT(12, 5) = 95040

Boomerang B
Boomerang C
Boomerang S
Boomerang D

We would like to Find all possible comb. of sample size 2, from population size = 4 (used later w/ central Limit Theorem)

Randomly select 2 of 4 to test. <u>order Does not matter</u>

$N = 4$
$n = 2$

$${}_4C_2 = \frac{4!}{2!(4-2)!} = \frac{\cancel{1} * \cancel{2} * 3 * 4}{1 * 2(1 * 2)} = \frac{12}{2} = 6$$

Excel = COMBIN(number, number_chosen)
= COMBIN(N, n)
= COMBIN(4, 2) = 6

Experimental outcomes = BC, BS, BD, CS, CD, DS

<u>order Does matter</u>

$${}_4P_2 = \frac{4!}{(4-2)!} = \frac{1 * 2 * 3 * 4}{1 * 2} = 12$$

Experimental outcomes = BC, BS, BD, CB, CS, CD, SB, SC, SD, DB, DC, DS

10 Methods of Assigning Probability

(a) Classical Method

All experimental outcomes are equally likely.

$$P(E_i) = \frac{\text{\# of favorable outcomes}}{\text{total \# of possible outcomes}}$$

Example: [1] Roll die, 6 equally likely outcomes

$$P(\text{Roll a } 6) = \frac{1}{6}$$

[2] Get Audited by district office, 3000 out of 3,000,000 Tax Returns

$$P(\text{Audit}) = \frac{3000}{3000000} = \frac{1}{1000}$$

(b) Relative Frequency Method

Law of Large Numbers; over a large number of Trials, Relative Frequency will approach True probability.

Use past Data to predict Future

or

Use past Data Available to estimate the proportion of the time the experimental outcome will occur if the Experiment is repeated a large number of times.

Example: instructor gave 10 A's out of 100 in past class. $P(A) = \frac{10}{100} = \frac{1}{10}$

(c) Subjective Method

[1] can't realistically assume outcomes equally likely

[2] Little Past Data exists

[3] You will use the best information you have, but it will be your personal belief.

Examples: who will win Super Bowl, merger between Google & Microsoft

11 Requirements for Assigning Probabilities (All 3)

Prob. between 0 & 1

$\boxed{1}$ $$0 \leq P(E_i) \leq 1 \quad \text{for all } i$$

E_i = i^{th} Experimental outcome

$P(E_i)$ = probability

Sum of All = 1

$\boxed{2}$ $$P(E_1) + P(E_2) + \dots + P(E_n) = 1$$

sum of the probabilities for <u>ALL</u> experimental outcomes must be one!

Example: Look back to page 4

12 **Event** = An event is a collection of sample points (1 or more)

*Note: sample points & events provide the foundation for the study of probability

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Experiment: Toss coin 3 times

Success = H

sample point = Experimental outcome

Total sample points	Toss 1	Toss 2	Toss 3	List of Sample points	# of H	Probability for each is equal
1	H	H	H	H, H, H	3	1/8 = .125
2	H	H	T	H, H, T	2	1/8 = .125
3	H	T	H	H, T, H	2	1/8 = .125
4	T	H	H	T, H, H	2	1/8 = .125
5	H	T	T	H, T, T	1	1/8 = .125
6	T	H	T	T, H, T	1	1/8 = .125
7	T	T	H	T, T, H	1	1/8 = .125
8	T	T	T	T, T, T	0	1/8 = .125
						Σ = 1

.125
.125
+ .125
= .375
or
3 * .125 = .375
or
$\frac{3}{8}$ = .375

Event = get 2 heads in 3 Flips = { 3 sample points match this requirement }

13 **Probability of an Event**

The probability of any event is equal to the sum of the probabilities of the sample points in the event. (Not always possible to list all sample points)

S = {(H, H, T), (H, T, H), (T, H, H)}

P (get 2 Heads in 3 Tries) = .125 P(HHT)
+ .125 P(HTH)
+ .125 P(THH)
.375

14 Because S, sample space, is an event, and it contains all sample points:

$$P(S) = 1$$

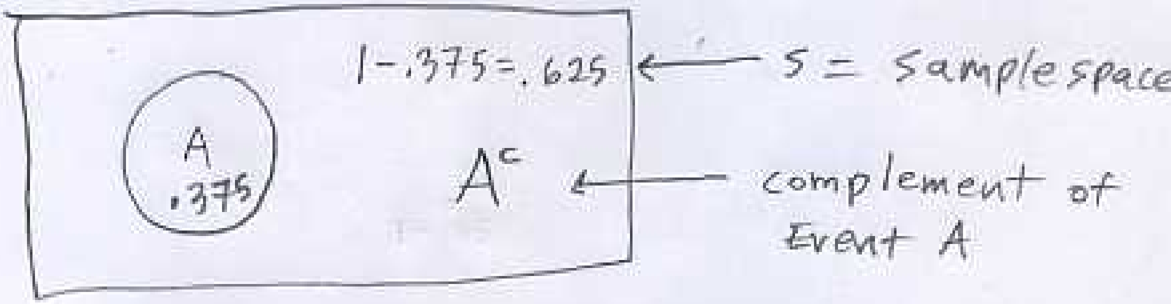
15 when the classical method is used to assign probability (all sample points are equally likely) $\Rightarrow$

$$P(E_i) = \frac{\text{\# of favorable Exp. outcomes}}{\text{total \# of All Exp. outcomes}}$$

16 Venn Diagram

Event A = Get 2 Heads in 3 Flips.

A^c = Not Event A (0, 1 or 3 rolls)



$P(S) = 1$

$P(A) = .375$

$P(A^c) = 1 - .375 = .625$

17 Complement Rule

$P(A^c) = 1 - P(A)$

or

$P(A) = 1 - P(A^c)$

"Dating only one person"

18 Mutually Exclusive

Two events are said to be mutually exclusive if the events have no sample points in common.

Example:

Event$_1$ = Roll odd number w/ 1 die

Event$_2$ = Roll even number w/ 1 die

1,3,5 2,4,6

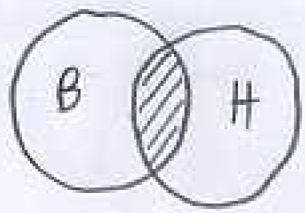
Sample Points Event 1 = 1, 3, 5

Sample Points Event 2 = 2, 4, 6

None in common. Therefore the events are mutually exclusive

$Event_B$ = have brown hair = B

$Event_H$ = have hazel eyes = H



Events are not mutually exclusive

19 Union

The union of B and H is the event containing all sample points belonging to B or H or Both

symbol

union = B or H or Both = $\cup$ = OR = "at least 1" = "1 or more"

20 Intersection

Given two events B & H, the intersection of B and H is the event containing the sample points belonging to both B and H.

symbol

Intersection = A and B = $\cap$ = AND = Both = Joint = concurrent = Both must be True

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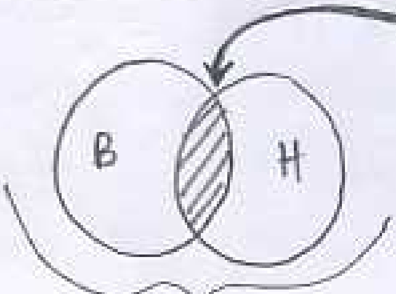
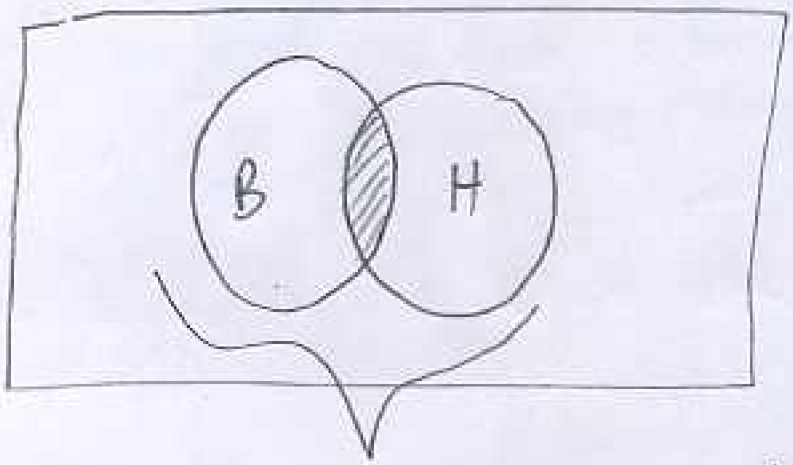
$P(B \cap H) = P(B \text{ AND } H)$ = Intersection



union = $P(B \cup H) = P(B \text{ OR } H)$ = Have Brown Hair or Hazel eyes or Both

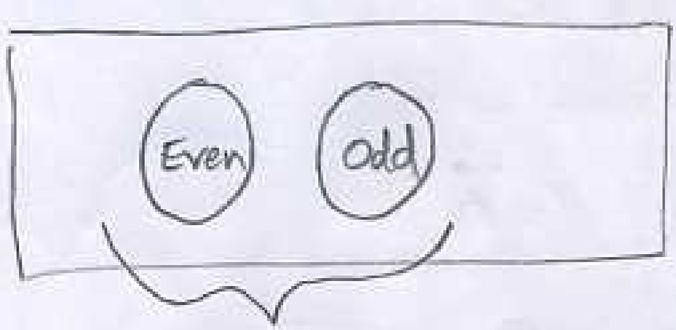
21 Addition Law for Probability



Not mutually Exclusive

$$P(B \cup H) = P(B \text{ or } H) = P(B) + P(H) - P(B \text{ and } H)$$

most subtract so we don't Double count!!!



Mutually Exclusive

$$P(E \cup O) = P(E \text{ or } O) = P(E) + P(O)$$

Example 1:

Flight Arrival		Frequency
Early	(E)	100
on time	(OT)	800
Late	(L)	75
canceled	(C)	25
		1000

Mutually Exclusive Events

$$P(E) = \frac{100}{1000} = .1$$

$$P(E \text{ or } OT) = P(E \cup OT) = \frac{(800+100)}{1000} = .9$$

Example 2:

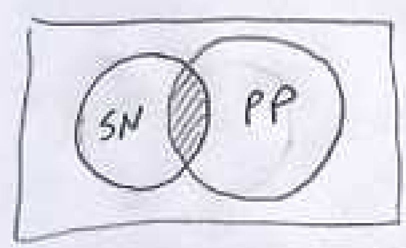
Travel survey of people who visit Seattle

visit Space Needle = SN
Visit Pike's Place Market = PP

Not M.E. events

$P(SN) = .45$
$P(PP) = .65$
$P(SN \text{ and } PP) = .25 \quad = P(SN \cap PP)$

$$P(SN \text{ or } PP) = .45 + .65 - .25$$



must subtract so we don't count twice!!

* Note problem may say: P(at least one site)
P(Either site)

22 Conditional Probability $P(A|B)$

"given that"

1. The probability of an event "given that" another event already occurred.

2. sample space has changed.

Example: Probability of pulling a Queen from a deck of cards (without replacement) affects the prob. of pulling the next card.

Event Q_1 = Pull Queen from 52 card deck
Event Q_2 = Pull 2nd Queen after 1st Queen Pulled

$$P(Q_1) = \frac{4}{52}$$

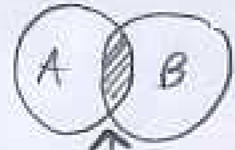
(51 cards left - sample space has changed)

$$P(Q_2|Q_1) = \frac{3}{51}$$

"given that"

1. conditional probability
2. Events are dependent
3. Events are not independent
4. sample space has changed
5. Q_1 is alread know to exist before we calculate $P(Q_2|Q_1)$

23 Joint Probability



$P(A \text{ and } B) = P(A \cap B) =$

Intersection
And
Joint
concurrent

Example:

	(M) Male	(F) Female	total
(P) promoted	288	36	324
Not Promoted (NP)	672	204	876
	960	240	1200

24 Joint Probability Table

Marginal Prob.
Joint Probabilities

	M	F	total
P	$\frac{288}{1200} = .24$	$\frac{36}{1200} = .03$	$.24 + .03 = .27$
NP	$\frac{672}{1200} = .56$	$\frac{204}{1200} = .17$	$.56 + .17 = .73$
total	$.24 + .56 = .8$	$.03 + .17 = .2$	$.27 + .73 = 1$ or $.8 + .2 = 1$

Joint

$P(M \text{ and } P) = .24$
$P(F \text{ and } P) = .03$
$P(M \text{ and } NP) = .56$
$P(F \text{ and } NP) = .17$

$$\left.\begin{aligned} P(M) &= \frac{960}{1200} = .8 \\ P(F) &= \frac{240}{1200} = .2 \end{aligned}\right\} \text{Marginal Prob.}$$

conditional Probabilities

$P(P \mid M) = \frac{288}{960} = .3$

$P(P \mid F) = \frac{36}{204} = .15$

does not prove discrimination But does support the argument presented by Females that there is discrimination

Probability Rules of Multiplication

Independent

25 ① Events are independent if the occurrence of one event does not affect the occurrence of another event

② sample space is not changed.

③ Rule of Independence $P(B) = P(B/A)$ or $P(A) = P(A|B)$ {A unaffected by occurrence of B

Example: ① Rolling a 6, does not affect the next roll

② A traffic jam today, generally does not affect whether you are in traffic jam tomorrow

③ whether or not Whole Foods Market stock price goes up does not affect whether or not Google's goes up.

Dependent

① occurrence of one event affects the occurrence of another event

② sample space is changed

Example: ① Probability of pulling Queen from a deck of cards (without replacement) affects the probability of pulling the next card.

(1st) $P(Q_1) = \frac{4}{52}$ ← depend on 1st pull

(2nd) $P(Q_2) = \frac{4}{51}$ or $\frac{3}{51}$ Depending on what

26 Multiplication Law for Probability

$$P(A \text{ and } B) = P(A \cap B) = P(B) * P(A|B)$$

or

$$P(A \text{ and } B) = P(A \cap B) = P(A) * P(B|A)$$

If events are independent:

since $P(B) = P(B/A)$

$P(A) = P(A|B)$

then

27 multiplication Law (Independence)

$$P(A \text{ and } B) = P(A \cap B) = P(A) * P(B)$$

Hint:

And $= \cap = *$

OR $= \cup = +$

Note: Two events with NON zero probabilities cannot be both M.E. → mutually exclusive & independent. Mutually exclusive means an event occurred & the other did not. Independent means the two exist but they are not related. M.E. event and other zero prob. event are dependent.