100	90's	80's	70's	60's	<60
1	4	5	12	3	7

Math 220
Spring 2024
Assessment 5
Dusty Wilson

Name: key

7:57
8:19

Sophie Germain proved to the world that even a woman can accomplish something in the most rigorous and abstract of sciences.
Carl Friedrich Gauss (1777-1855)
German mathematician

No work = no credit

1. Warm-ups

   (a) (1 point) $\vec{e}_1^T \vec{e}_1$ $\quad \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1]$

   (b) (1 point) $\vec{e}_1 \, \vec{e}_2^T$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

   (c) (1 point) If $A_{n\times m}$ then rank+nullity = m

2. (1 point) Based upon the quote by Gauss (above), name a famous female mathematician? Answer using complete English sentences.

   Sophie Germain was a rad female mathematician

3. (4 points) Find the $\mathcal{B}$-coordinates of the vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ given the $\mathcal{B}$ basis $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}\}$

   ~~$P_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$~~

   ~~$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_{\mathcal{B}}$~~

4. (4 points) The set $\mathcal{B} = \{1+2t, 3+5t\}$ is a basis for $\mathbb{P}_1$. Find the coordinate vector of $p(t) = 1+t$ relative to $\mathcal{B}$.

   $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $P_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $[p]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

   so $p(t) = -2(1+2t) + 1(3+5t)$

   either is ok

5. (2 points) The subspace $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$. List all possible value(s) for $\dim(H)$.

   $\dim(H) = 0, 1, 2$

6. (7 points) Consider matrix $C = \begin{bmatrix} 1 & 0 & 3 & 4 & 2 & -1 \\ 2 & 0 & 1 & 3 & -1 & 3 \\ 3 & 0 & 4 & 7 & +1 & 2 \\ 4 & 0 & 1 & 5 & -3 & 7 \end{bmatrix}$.

(a) (2 points) Find a basis for the column space of C

$$C \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for $\text{col}(C) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} \right\}$

(b) (1 point) The rank of C

2

(c) (2 points) The null space of C

$\text{nul}(C) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) (1 point) A basis for row(C)

basis for $\text{row}(C) = \{(1,0,0,1,-1,2), (0,0,1,1,1,-1)\}$

(e) (1 point) Choose any non-zero $\vec{u} \in \text{nul}(C)$ and $\vec{v} \in \text{row}(C)$ and then find $\vec{v}\,\vec{u}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

7. (2 points) True or False. A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$. Justify your answer.

False. Ex: $z=1$ isn't a subspace because it doesn't include $\vec{0}$.

8. (4 points) Prove (or disprove) the Unique Representation Theorem.

Claim: Let $B = \{\vec{b}_1, ..., \vec{b}_n\}$ be a basis for a vector space V. Then for each $\vec{x} \in V$, there exists a unique set of scalars $c_1, ..., c_n$ such that $\vec{x} = c_1\vec{b}_1 + ... + c_n\vec{b}_n$.

proof.

Let basis B and vector $\vec{x} \in V$ be given.

Suppose $\vec{x}$ has two representations

$\vec{x} = c_1\vec{b}_1 + ... + c_n\vec{b}_n$ and $\vec{x} = d_1\vec{b}_1 + ... + d_n\vec{b}_n$

$\Rightarrow \vec{x} - \vec{x} = \vec{0} = c_1\vec{b}_1 + ... + c_n\vec{b}_n - d_1\vec{b}_1 - ... - d_n\vec{b}_n$

$= (c_1 - d_1)\vec{b}_1 + ... + (c_n - d_n)\vec{b}_n$

Since B is a basis, its vectors are linearly independent & the homogeneous equation has only the trivial solution,

$\Rightarrow c_1 - d_1 = ... = c_n - d_n = 0$

or $c_1 = d_1 \quad ... \quad c_n = d_n$

$\Rightarrow\Leftarrow$

$\therefore \vec{x}$ has a unique representation.

Q.E.D.

9. (6 points) Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$.

(a) (2 points) If $[\vec{y}]_\mathcal{B} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, find $\vec{y}$

$P_\mathcal{B} = \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}$

$\vec{y} = P_\mathcal{B}[\vec{y}]_\mathcal{B} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

(b) (2 points) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$

$\left[\begin{array}{cc|cc} \overset{B}{2} & 0 & \overset{C}{3} & -1 \\ 2 & 4 & -2 & 7 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & -5/4 & 2 \end{array}\right]$ with $\underbrace{\begin{matrix} 3/2 & -1/2 \\ -5/4 & 2 \end{matrix}}_{P_{\mathcal{B}\leftarrow\mathcal{C}}}$

AND $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \frac{1}{19}\begin{bmatrix} 16 & 4 \\ 10 & 12 \end{bmatrix}$

(c) (2 points) Find the $\mathcal{C}$-coordinates of $[\vec{x}]_\mathcal{B} = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$

$\begin{bmatrix} 3/2 & -1/2 \\ -5/4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -7 \\ 9 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}_C$