100	90's	80's	70's	60's	<60
2	4	3	7	0	5

Name: Key

Math 220
Winter 2024
Assessment 4
Dusty Wilson

high = 100
mean = 76.6%
med = 78.4%

What is real? How do you define 'real'? If you're talking about what you can feel, what you can smell, what you can taste and see, then 'real' is simply electrical signals interpreted by your brain.
Morpheus in *The Matrix* (1999)

No work = no credit

1. Warm-ups

   (a) (1 point) $AA^{-1} = I$

   (b) (1 point) $\vec{e}_2\,\vec{e}_2^{\,T} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

   (c) (1 point) $\vec{e}_2^{\,T}\vec{e}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

2. (2 points) In light of the quote by Morpheus, what is real? Answer using complete English sentences.

   This is a tough question. I am going w/ Plato and claiming that everything is simply a shadow of a greater abstract ideal reality. I don't know that I fully believe this, but at least I can articulate a position.

3. (4 points) Calculate $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$. Is this determinant invertible? Why or why not.

   $$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5$$

   Determinants are not invertible so the question is nonsense.

4. (4 points) Find the determinant of $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Is A invertible? Why or why not.

   $$\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - \underbrace{1\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix}}_{0} + 1\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$$

   $$= 2 + (-4)$$

   $$= -2.$$

   Since $|A| \neq 0$, A is invertible.

5. (4 points) Find the determinant of $B = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -5 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$. Is B invertible? Why or why not.

$$|B| = -5 \begin{vmatrix} 1 & 2 & 7 \\ 2 & 5 & 4 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -5(1) \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}$$

$$= -5$$

6. (7 points) For the matrix $A_{n\times n}$, there are at least 13 statements equivalent to, "A is invertible." List at least seven of them. List more for extra credit (2 points max).

i.) A is invertible	vi.) The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.
ii.) $A \sim I_{n\times n}$	vii.) The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$
iii.) A has n pivot positions	viii.) The columns of A span $\mathbb{R}^n$
iv.) $A\vec{x} = \vec{0}$ has only the trivial soln.	xi.) (1 pt extra credit) The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
v.) The columns of A form a linearly independent set.	x.) (1 pt extra credit) $\det A \neq 0$

There is $C_{n\times n}$ such that $CA = I$

There is a $D_{n\times n}$ such that $AD = I$.

What is your name (for the calculator portion): Mr. key.

7. (2 points) True or False: The determinant of A is the product of the diagonal entries in A. Explain.

False. For example $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ while the product of the diagonal entries is 1.

8. (4 points) One interpretation of the determinant is as the scaling factor of a linear transformation. For example, "A region with area X becomes a region of area $X * det(A)$ under the linear transformation $T(\vec{x}) = A\vec{x}$"

If $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, (a.) find and (b.) interpret the determinants of A and A^{-1} using the language of scaling factors.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -2 & 1 \end{bmatrix}$$

(a) $|A| = 2$ and $|A^{-1}| = \frac{1}{2}$

(b) A region doubles in area under $T(\vec{x}) = A\vec{x}$, but then halves under the inverse getting you back where you started.

9. (4 points) Given the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ and vector $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, solve the matrix equation $A\vec{x} = \vec{b}$ using the matrix inverse. You may use a calculator, but show enough work so that it is clear that you could do this by hand if necessary.

$$\text{rref}([A|I]) \Rightarrow A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

And $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

10. (4 points) Prove the following.

Claim: If A is an invertible $n \times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

proof.

(Existence) Let $\vec{b} \in \mathbb{R}^n$ and invertible $A_{n \times n}$ be given.

solve $A\vec{x} = \vec{b} \Leftrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \Leftrightarrow I\vec{x} = A^{-1}\vec{b}$

$\therefore A\vec{x} = \vec{b}$ has a solution $\vec{x} = A^{-1}\vec{b}$.

(uniqueness). Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be solutions to $A\vec{x} = \vec{b}$, where A is invertible

$\Rightarrow A\vec{u} = A\vec{v} = \vec{b}$

$\Rightarrow A^{-1}A\vec{u} = A^{-1}A\vec{v} = A^{-1}\vec{b}$

$\Rightarrow I\vec{u} = I\vec{v} = A^{-1}\vec{b}$

$\Rightarrow \vec{u} = \vec{v}$

$\therefore$ The solution is unique.

Thus if A is invertible, then for each $\vec{b} \in \mathbb{R}^n$ the equation $A\vec{x} = \vec{b}$ has the unique soln $\vec{x} = A^{-1}\vec{b}$.