Math 220
Spring 2024
Assessment 3
Dusty Wilson

Name: Key

When a philosopher says something that is true then it is trivial. When he says something that is not trivial then it is false.
Carl Friedrich Gauss (1777-1855)
German mathematician

No work = no credit

1. Warm-ups

   (a) (1 point) I^2 $= I$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) (1 point) $\vec{e}_1\,\vec{e}_2^{\,T}$ $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

   (c) (1 point) $\vec{e}_1^{\,T}\vec{e}_1$ $= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$ or 1

2. (1 point) Based upon the quote by Gauss (above), what do you think Gauss' view of philosophers was? Answer using complete English sentences.

   Gauss seems to have held philosophers in low regard.

3. (8 points) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$, find the following:

   (a) (2 points) $2A + B$

   undefined, mismatched dimensions.

   (b) (2 points) $A^T + B$

   $$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \\ 4 & 9 \end{bmatrix}$$

   (c) (2 points) AB

   $$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 14 \\ 7 & 25 \end{bmatrix}$$

   (d) (2 points) BA

   $$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 9 \\ 2 & 6 & 12 \\ 4 & 11 & 21 \end{bmatrix}$$

4. (7 points) For the matrix $A_{n\times n}$, there are at least 13 statements equivalent to, "A is invertible." List at least seven of them. List more for extra credit (2 points max).

i.) A is invertible	vi.) The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one
ii.) $A \sim I_{n\times n}$	vii.) The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$.
iii.) A has n pivot positions	viii.) The columns of A span $\mathbb{R}^n$
iv.) $A\vec{x} = \vec{0}$ has only the trivial solution.	xi.) (1 pt extra credit) The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
v.) The columns of A form a linearly independent set.	x.) (1 pt extra credit) $\det A \neq 0$

There is a $C_{n\times n}$ s.t. $CA = I$

There is a $D_{n\times n}$ such that $AD = I$.

5. (4 points) Given the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ and vector $\vec{b} = \begin{bmatrix} 4 \\ 7 \\ 11 \end{bmatrix}$, solve the matrix equation $A\vec{x} = \vec{b}$ using the matrix inverse. You may use a calculator, but show enough work so that it is clear that you could do this by hand if necessary.

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$\Rightarrow x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

6. (4 points) Explain the process for finding the inverse of an $n \times n$ matrix A. or determining if A is singular.

row reduce $[A \mid I_{n\times n}]$

(case 1) $\sim [I \mid B]$ where $B = A^{-1}$

(case 2) $\sim [\text{not } I \mid C]$ A is singular.

7. (2 points) True or False. If the equation $A_{n\times n}\vec{x} = \vec{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix. Justify your answer.

Both of these are equiv statements in the invertible matrix theorem.

8. (4 points) Prove the following (without reference to the invertible matrix theorem).

Claim: If A is an invertible $n \times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

proof.

Let invertible $n\times n$ A and $\vec{b} \in \mathbb{R}^n$ be given.

(existence) Suppose $A\vec{x} = \vec{b}$

$\Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\Rightarrow I\vec{x} = A^{-1}\vec{b}$

$\Rightarrow \vec{x} = A^{-1}\vec{b}$

But could it have other solutions?

Thus a solution to $A\vec{x} = \vec{b}$ has $\vec{x} = A^{-1}\vec{b}$ as a solution.

(uniqueness) Suppose $A\vec{x} = \vec{b}$ had two solutions $\vec{u}, \vec{v} \in \mathbb{R}^n$

$\Rightarrow A\vec{u} = \vec{b}$ and $A\vec{v} = \vec{b}$

$\Rightarrow A\vec{u} = A\vec{v}$

$\Rightarrow A^{-1}A\vec{u} = A^{-1}A\vec{v}$

$\Rightarrow I\vec{u} = I\vec{v}$

$\Rightarrow \vec{u} = \vec{v}$

Thus the solution is unique.

$\therefore$ $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$