5.2: The Characteristic Equation

Math 220: Linear Algebra

To find eigenvalues of a square matrix, we are finding non-trivial solutions to the equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$. By the invertible matrix theorem, this is the same as finding λ such that $A-\lambda I$ is singular / not invertible. But this occurs when the determinant is zero.

Ex 1: Find the Eigenvalues of $A=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$.

solve $\underbrace{(A-\lambda I)}_{\begin{bmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{bmatrix}}\vec{x}=\vec{0}$

which means we must

solve $\det\left(\begin{bmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{bmatrix}\right)=0$

$$\Rightarrow 0 = \begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}$$
$$= (5-\lambda)^2-9$$
$$= 25-10\lambda+\lambda^2-9$$
$$= \lambda^2-10\lambda+16$$
$$= (\lambda-8)(\lambda-2)$$

so there are two eigenvalues $\lambda=2,8$.

Theorem The Invertible Matrix Theorem (continued)
Let A be an $n\times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A. ← we saw this in the previous solution,

t. The determinant of A is *not* zero.

recall from chapter 3

Theorem 3 Properties of Determinants

Let A and B be $n \times n$ matrices.

a. A is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

This is particularly helpful finding eigenvalues.

d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

We can now determine when the matrix $A - \lambda I$ is not invertible by solving the characteristic equation, $\det(A - \lambda I) = 0$.

Ex 2: Find the characteristic equation of $A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$ and the eigenvalues of A.

solve $0 = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{vmatrix}$

$= (4-\lambda)(3-\lambda)(2-\lambda)$.

$\Rightarrow \lambda = 4, 3, 2$

Triangular matrix

Ex 3: Find the characteristic equation of $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -1 & 2 & 3 & 0 \\ 5 & 0 & 1 & -1 \end{bmatrix}$ and eigenvalues.

The characteristic equation is

$0 = (4-\lambda)(3-\lambda)^2(-1-\lambda)$

The eigenvalues are

$\lambda = 4$, $\lambda = 3$ (algebraic multiplicity 2), $\lambda = -1$

If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the characteristic polynomial of A.

The eigenvalue of 3 in Ex 3. is said to have (algebraic) multiplicity 2 because the factor $3-\lambda$ occurs twice in the characteristic polynomial.

Ex 4: The Characteristic polynomial of a 7×7 matrix is $\lambda^7 - 8\lambda^5 + 16\lambda^3$. Find the eigenvalues and their multiplicities.

Solve $0 = \lambda^7 - 8\lambda^5 + 16\lambda^3$

$= \lambda^3(\lambda^4 - 8\lambda^2 + 16)$

$= \lambda^3(\lambda^2 - 4)^2$

$= \lambda^3(\lambda - 2)^2(\lambda + 2)^2$

Eigenvalues	Multiplicity
$\lambda = 0$	3
$\lambda = 2$	2
$\lambda = -2$	2

Similarity we will use this concept in a subsequent section when we study diagonalization.

Two $n \times n$ matrices *A* and *B* are considered similar if there is an invertible matrix *P* such that

$$P^{-1}AP = B \qquad \text{Or} \qquad A = PBP^{-1}$$

It doesn't matter if the inverse is 1st or 2nd.

We can also write *Q* for P^{-1} and get

$$QAQ^{-1} = B \qquad \text{Or} \qquad A = Q^{-1}BQ$$

vocab: Changing *A* into $P^{-1}AP$ is called the similarity transformation.

Theorem 4
If $n \times n$ matrices *A* and *B* are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof: Let similar A and B be given.

$\Rightarrow \exists$ invertible P s.t. $B = P^{-1}AP$

$\Rightarrow B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P$

$$\Rightarrow \det(B - \lambda I) = \det\left(P^{-1}(A - \lambda I)P\right)$$
$$= \det(P^{-1})\det(A - \lambda I)\det(P)$$
$$= \det(A - \lambda I) \text{ since } \det(P^{-1}) = \frac{1}{\det(P)}$$

$\therefore$ The characteristic polynomials and eigenvalues are the same.

Warnings:

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If *A* is row equivalent to *B*, then $B = EA$ for some invertible matrix *E*.) Row operations on a matrix usually change its eigenvalues.

Practice Problem

Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

solve $0 = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{vmatrix}$

$= (1-\lambda)(2-\lambda) + 16$

$= 2 - 3\lambda + \lambda^2 + 16$

$\Rightarrow \quad 0 = \lambda^2 - 3\lambda + 18$ characteristic equation

$\Rightarrow \lambda = \dfrac{3 \pm \sqrt{9 - 4(1)(18)}}{2(1)}$

$= \dfrac{3 \pm \sqrt{-63}}{2}$

$= \dfrac{3 \pm 3i\sqrt{7}}{2}$ eigenvalues (all complex)