5.1: Eigenvectors and Eigenvalues

Math 220: Linear Algebra

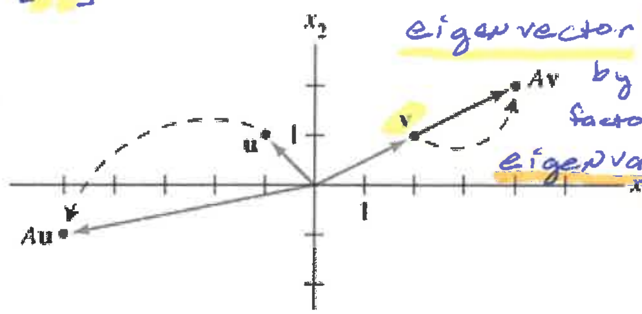
Ex 1: Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Calculate $A\mathbf{u}$ and $A\mathbf{v}$.

What do you notice about either of them?

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

notice that A scales the eigenvector $\vec{v}$ by a factor of eigenvalue 2.



notice that the special property $A\vec{v} = \lambda\vec{v}$ w/ $\lambda = 2$ is satisfied.

Definition
An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathbf{x} = \lambda\mathbf{x}$; such an **x** is called an *eigenvector corresponding to* λ.

Ex 2: Is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? If so, find the eigenvalue.

check: $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Yes and $\lambda = 4$.

Is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? If so, find the eigenvalue.

check: $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix} \neq \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ so not an eigenvector.

Ex 3: Show that 5 is an eigenvalue of the matrix $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, and find the corresponding eigenvector.

solve $A\vec{x} = 5\vec{x}$

$\Rightarrow A\vec{x} - 5\vec{x} = \vec{0}$

$\Rightarrow A\vec{x} - 5I\vec{x} = \vec{0}$

$\Rightarrow (A - 5I)\vec{x} = 0$

we need the null space of $A - 5I$.

$\to [A - 5I \mid \vec{0}] = \begin{bmatrix} -4 & 2 & 0 \\ 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\to \vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

The eigenvector corresponding to $\lambda = 5$ is $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The eigenvector must be non-zero, but an eigenvalue may be zero.

So λ is an eigenvalue of an $n \times n$ matrix, if and only if

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

What would another name for the solutions to this equation be?

The solutions are the nullspace of $(A - \lambda I)$

But we already know that any null space is a subspace of $\mathbb{R}^n$, so we call it the eigenspace of *A*.

Ex 4: Find a basis for the eigenspace given $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$, $\lambda = 3$

$(A - 3I)\vec{x} = \vec{0}$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\to$ null space of $A - 3I$ = eigenspace corresponding to $\lambda = 3$ ← 1 eigenvalue

notice

$= \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

AND

This has a basis of eigenvectors $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ ← 2 L.I eigenvectors

Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Ex 5: Find the eigenvalues of $\begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$\lambda = 0, 2, 3$

What does it mean for a matrix A to have an eigenvalue of 0?

$A\vec{x} = 0\vec{x} = \vec{0}$ there are non-trivial solutions to the homogeneous equation. $A\vec{x} = \vec{0}$.

This means that 0 is an eigenvalue of A if and only if A is not invertible.

This will be added to our invertible matrix theorem in 5.2.

Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Proof: (by contradiction)

Suppose $\vec{v}_1, \ldots, \vec{v}_r$ are linearly dependent and $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i = 1, \ldots, r$ w/ λ's distinct.

$\Rightarrow$ there is a $\vec{v}_{p+1} = c_1\vec{v}_1 + \ldots + c_p\vec{v}_p$ (*) w/ not all $c_1, \ldots, c_p = 0$, And $\vec{v}_1, \ldots, \vec{v}_p$ linearly independent.

$\Rightarrow A\vec{v}_{p+1} = A(c_1\vec{v}_1 + \ldots + c_p\vec{v}_p)$

$\Rightarrow \lambda_{p+1}\vec{v}_{p+1} = c_1 A\vec{v}_1 + \ldots + c_p A\vec{v}_p$

$\Rightarrow \lambda_{p+1}\vec{v}_{p+1} = c_1\lambda_1\vec{v}_1 + \ldots + c_p\lambda_p\vec{v}_p$

And $-\lambda_{p+1} v_{p+1} = -c_1\lambda_{p+1}\vec{v}_1 + \ldots + -c_p\lambda_{p+1}\vec{v}_p$ by multiplying equation (*) by "$-\lambda_{p+1}$"

Now we add: $0 = c_1\underbrace{(\lambda_1 - \lambda_{p+1})}_{\neq 0}\vec{v}_1 + \ldots + c_p\underbrace{(\lambda_p - \lambda_{p+1})}_{\neq 0}\vec{v}_p$

since λ's distinct

Since $\vec{v}_1, \ldots, \vec{v}_p$ are L.I. this implies $c_1 = \ldots = c_p = 0 \Rightarrow\Leftarrow$

$\therefore \vec{v}_1, \ldots, \vec{v}_r$ are linearly independent.

Practice Problems

1. Is 5 an eigenvalue of $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?

solve $(A - 5I)\vec{x} = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

not actually an eigenspace since empty

no free variables and so no non-zero vectors in the null space/eigenspace.

2. If **x** is an eigenvector of A corresponding to λ, what is $A^3\mathbf{x}$?

$A\vec{x} = \lambda\vec{x}$

$\Rightarrow A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda A\vec{x} = \lambda^2\vec{x}$

$\Rightarrow A^3\vec{x} = A(A^2\vec{x}) = A(\lambda^2\vec{x}) = \lambda^2 Ax = \lambda^3\vec{x}$

4. If A is an $n \times n$ matrix and λ is an eigenvalue of A, show that 2λ is an eigenvalue of $2A$.

suppose A has eigenvector $\vec{x}$ w/ eigenvalue λ

$\Rightarrow A\vec{x} = \lambda\vec{x}$

$\Rightarrow (2A)\vec{x} = 2(A\vec{x}) = 2(\lambda\vec{x}) = (2\lambda)\vec{x}$