2.3: Characteristics of Invertible Matrices

Math 220: Linear Algebra

Theorem 8 The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Theorem 5 from 2.2 could also make g. state $A\vec{x} = \vec{b}$ has a unique solution.

If A and B are square matrices, and $AB = I$, then by j. and k. both A and B are invertible with $B = A^{-1}$ and $A = B^{-1}$.

proof.

Let square matrices A & B be given. with $AB = I$.

$\Rightarrow A$ is invertible (by k).

$\Rightarrow A^{-1}AB = A^{-1}I \Rightarrow IB = A^{-1} \Rightarrow B = A^{-1}$

And B is invertible (by J)

$\Rightarrow ABB^{-1} = IB^{-1} \Rightarrow AI = B^{-1} \Rightarrow A = B^{-1}$

Q.E.D.

2.3: Characteristics of Invertible Matrices

The Invertible Matrix Theorem essentially divides the set of all $n \times n$ matrices into two disjoint classes:

Invertible (A has an Inverse)

- non-singular
- n pivot positions
- columns of A are L.I.
- $A\vec{x} = \vec{0}$ has only the triv. sol.
- LT is onto $\mathbb{R}^n$
- cols of A span $\mathbb{R}^n$
- A^T is invertible
- $A\vec{x} = \vec{b}$ has a soln $\forall \vec{x} \in \mathbb{R}^n$.
- L.T. is 1-1
- A is row equivalent to I.

Not invertible (A doesn't have an inverse)

- singular
- $< n$ pivot positions
- columns of A are L.D.
- $A\vec{x} = \vec{0}$ has non-trivial solutions
- L.T. is not onto $\mathbb{R}^n$
- cols of A do not span $\mathbb{R}^n$
- A^T is not invertible.
- $\exists\, \vec{b} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$ does not have a solution
- L.T. is not 1-1
- A is not row equivalent to I.

Ex 1: Use the Invertible Matrix Theorem to determine if the following are invertible.

use the calculator

$$A = \begin{bmatrix} 5 & 2 & 3 \\ 7 & 1 & 2 \\ 11 & -3 & 6 \end{bmatrix}$$

rref([A ⁝ I])

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -3/19 & 21/76 & -1/76 \\ 0 & 1 & 0 & 5/19 & 3/76 & -11/76 \\ 0 & 0 & 1 & 8/19 & -37/76 & 9/76 \end{array}\right]$$

3 pivots so A is invertible.

$$A^{-1} = \frac{1}{76}\begin{bmatrix} -12 & 21 & -1 \\ 20 & 3 & -11 \\ 32 & -37 & 9 \end{bmatrix}$$

by hand.

$$B = \begin{bmatrix} 1 & -3 & -2 \\ 5 & -1 & 18 \\ 4 & 2 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -2 & 1 & 0 & 0 \\ 5 & -1 & 18 & 0 & 1 & 0 \\ 4 & 2 & 20 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 5R_1 \to R_2$; $R_3 - 4R_1 \to R_3$

$$\sim \begin{bmatrix} 1 & -3 & -2 & 1 & 0 & 0 \\ 0 & 14 & 28 & -5 & 1 & 0 \\ 0 & 14 & 28 & -4 & 0 & 1 \end{bmatrix}$$

$R_2 - R_3 \to R_3$

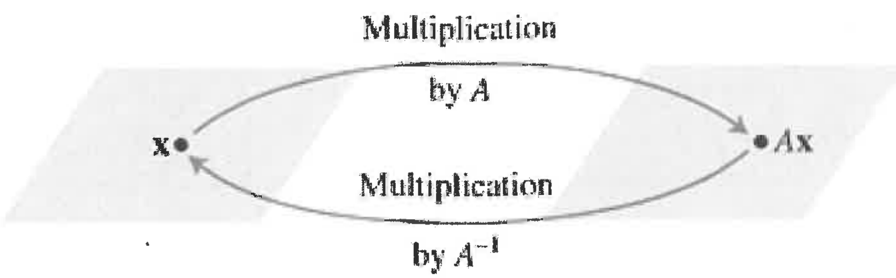
$$\sim \begin{bmatrix} 1 & -3 & -2 & 1 & 0 & 0 \\ 0 & 14 & 28 & -5 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

2 pivots so A is not invertible.

2.3: Characteristics of Invertible Matrices

Be careful, the Invertible Matrix Theorem only applies to square matrices.

If A is invertible, we can also think about $A^{-1}A\vec{x} = \vec{x}$ in light of linear transformations.



In general, a Linear Transformation $T:\mathbb{R}^N \to \mathbb{R}^N$ is invertible if there exists a function $S:\mathbb{R}^N \to \mathbb{R}^N$ such that

$$S\left(T(\vec{x})\right)=\vec{x} \quad \text{for all } \vec{x}\in \mathbb{R}^N$$

$$T\left(S(\vec{x})\right)=\vec{x} \quad \text{for all } \vec{x}\in \mathbb{R}^N$$

We call *S* the inverse of *T* and write it as T^{-1}.

Theorem 9

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let *A* be the standard matrix for *T*. Then *T* is invertible if and only if *A* is an invertible matrix. In that case, the linear transformation *S* given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

Ex 2: What can be said about a one-to-one linear transformation $T:\mathbb{R}^N \to \mathbb{R}^N$?

T is 1-1 $\iff$ cols of A span $\mathbb{R}^N$ (by Thm 12b in 1.9)

$\iff$ A is invertible (Thm 8 in 2.3)

$\iff$ T is invertible (Thm 9 in 2.3)

Practice Problems

2. Suppose that for a certain $n \times n$ matrix *A*, statement (g) of the Invertible Matrix Theorem is *not* true. What can you say about equations of the form $A\mathbf{x} = \mathbf{b}$?

$\exists \vec{b} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$ has no soln.

3. Suppose that *A* and *B* are $n \times n$ matrices and the equation $AB\mathbf{x} = \mathbf{0}$ has a nontrivial solution. What can you say about the matrix *AB*?

(AB) is not invertible.