2.1: Matrix Operations

Math 220: Linear Algebra

If A is an $m \times n$ matrix with m rows and n columns, then the entry in the ith row and jth column is denoted by a_{ij} and is called the (i, j)-entry.

$$\begin{array}{cc} & \text{Column } j \\ \text{Row } i & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \\ & \begin{array}{ccccc} \uparrow & \quad & \uparrow & \quad & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array} \end{array}$$

The diagonal entries are $a_{11}, a_{22}, a_{33}, \ldots$ and they form the main diagonal.

A diagonal matrix is a square matrix ($n \times n$) whose non-diagonal entries are all zero. The identity matrix I_n is a diagonal matrix with 1's down the diagonal.

The zero matrix has all zeros in all of its entries and is written just as 0.

Two matrices are equal if they are the same size and the corresponding entries are equal.

The sum of two matrices A + B is the sum of their corresponding entries. Thus, two matrices can only be added if their size (m×n) is the same. Otherwise, the sum is not defined.

Ex 1: Given $A = \begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

Find the following, if defined.

a) $A+B = \begin{bmatrix} 2+1 & -1+2 & 0+3 \\ -3+4 & 3+5 & -2+6 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 & 3 \\ 1 & 8 & 4 \end{bmatrix}$

b) $B+C$ Not defined because the dimensions don't match.

The scalar multiple rA is the matrix whose entries are r times each entry of A.

The matrix $-A$ represents $(-1)A$ and $A-B$ is the same as $A+(-1)B$.

Ex 2: Given $A=\begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find

a) $2A = \begin{bmatrix} 4 & -2 & 0 \\ -6 & 6 & -4 \end{bmatrix}$

b) $B-2A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 0 \\ -6 & 6 & -4 \end{bmatrix}$

$= \begin{bmatrix} -3 & 4 & 3 \\ 10 & -1 & 10 \end{bmatrix}$

Theorem 1
Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A+B=B+A$

b. $(A+B)+C=A+(B+C)$

c. $A+0=A$

d. $r(A+B)=rA+rB$

e. $(r+s)A=rA+sA$

f. $r(sA)=(rs)A$

Matrix Multiplication

Definition
If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \cdots \ \ \mathbf{b}_p] = [A\mathbf{b}_1 \ \ A\mathbf{b}_2 \ \ \cdots \ \ A\mathbf{b}_p]$$

Ex 3: Given $A=\begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, compute CA.

$C\mathbf{a}_1 = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $C\mathbf{a}_2 = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ $C\mathbf{a}_3 = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$

AND $\underset{2\times 2}{C} \cdot \underset{2\times 3}{A} = \begin{bmatrix} -1 & 5 & -6 \\ 1 & 1 & -2 \end{bmatrix}_{3\times 2}$

Ex 4: Given $A=\begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, is the matrix AC defined?

$A_{2\times 3}\ C_{2\times 2}$

No, the dimensions are mismatched.

Row–Column Rule for Computing *AB*

If the product *AB* is defined, then the entry in row *i* and column *j* of *AB* is the sum of the products of corresponding entries from row *i* of *A* and column *j* of *B*. If $(AB)_{ij}$ denotes the (i,j)-entry in *AB*, and if *A* is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

(Row i of A) "*" (column j of B)

dot product

Ex 5: Find the entries of the 3rd row of *AB*, where $A=\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}$, $B=\begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$

$A_{4\times 3}\ B_{3\times 2} = \begin{bmatrix} \square & \square \\ \square & \square \\ -53 & -58 \\ \square & \square \end{bmatrix}$

$-58 = 6\cdot(-6) + -8\cdot 1 + (-7)\cdot 2$

$-53 = 6\cdot 4 + (-8)\cdot 7 + (-7)\cdot 3$

We could have just ignored the rest of *A* and computed $\begin{bmatrix} 6 & -8 & -7 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$

$$\boxed{\text{row}_i(AB) = \text{row}_i(A)\cdot B}$$

Theorem 2

Let *A* be an $m \times n$ matrix, and let *B* and *C* have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left distributive law)

c. $(B+C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_mA = A = AI_n$ (identity for matrix multiplication)

While the following properties are all true, be careful, the commutative property is not true, that is, $AB \neq BA$.

Ex 6: Let $A=\begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$. Show that these two matrices do not commute. That is, verify that $AB \neq BA$.

$$AB = \begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ -5 & -23 \end{bmatrix}$$

not equal.

$$BA = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} -10 & 7 \\ 14 & -12 \end{bmatrix}$$

Warnings:

1. In general, $AB \neq BA$.

2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)

3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

10. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$.

Verify that $AB = AC$ and yet $B \neq C$.

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

Not equal

equal

$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

12. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

$$\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If A is an $n \times n$ matrix and if k is a positive integer, then $A^k = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$

Given an $m \times n$ matrix A, then the transpose of A is the $n \times m$ matrix, denoted by A^T whose columns are formed by the corresponding rows of A.

Ex 7: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 2 & 4 \\ 6 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -4 & -5 \end{bmatrix}$. Find

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad B^T = \begin{bmatrix} 1 & 5 & 2 & 6 \\ 3 & 7 & 4 & 8 \end{bmatrix}$ $\quad C^T = \begin{bmatrix} 2 & -3 \\ 1 & -4 \\ 0 & -5 \end{bmatrix}$

Theorem 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ $\leftarrow$ This will be important in ch6 w/ orthogonal matrices

Practice Problems

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \textbf{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

$$(A\vec{x})^T = \left(\begin{bmatrix} -4 \\ 2 \end{bmatrix}\right)^T = \begin{bmatrix} -4 & 2 \end{bmatrix}$$

same

$$\vec{x}^T A^T = \underset{1\times 2}{\begin{bmatrix} 5 & 3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}} = \begin{bmatrix} -4 & 2 \end{bmatrix}$$

$$\vec{x}^T\vec{x} = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 34 \end{bmatrix}$$

$$\vec{x}\vec{x}^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 25 \end{bmatrix}$$

$A^T_{2\times 2}\, x^T_{1\times 2}$ mismatched dimensions and so undefined.

2. Let A be a 4×4 matrix and let **x** be a vector in $\mathbb{R}^4$. What is the fastest way to compute $A^2\mathbf{x}$? Count the multiplications.

$(A \cdot A)\vec{x}$

$(A \cdot A)$: 16 entries, 4 mult. each $= 64$

$A^2\vec{x}$: 4 entries, 4 mult. each $= 16$

There are a total of: 80 multiplications

$A(\underbrace{A\vec{x}}_{\vec{v}})$

$\vec{v}$ 4 entries, 4 mult. each $= 16$

$A(A\vec{x}) = A\vec{v}$ also 16 multiplies.

There are a total of 32 multiplications.

3. Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB?

AB is an $m \times p$ matrix all of whose entries are identical.