# 1.4: Matrix Equations

## Math 220: Linear Algebra

### Definition

If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of $A$ and $\mathbf{x}$**, denoted by $A\mathbf{x}$, **is the linear combination of the columns of $A$ using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \cdots \ \ \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

$A\mathbf{x}$ is only defined if the number of <u>columns</u> of A equals the number of <u>entries</u> in $\mathbf{x}$.

$A_{m \times p} \cdot \vec{x}_{p \times 1} \leftarrow$ product is $m \times 1$

**Ex 1:** (A is 2x3)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$\uparrow A \qquad \uparrow \vec{x} \qquad \uparrow \vec{b}$

(A is 3x2)

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$\uparrow A \qquad \uparrow \vec{x} \qquad \uparrow \vec{b}$

# 1.4: Matrix Equations

**Ex 2:** For $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^3$ Write the linear combination of $5\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$ as a matrix times a vector.

$$5\vec{u}_1 - 1\vec{u}_2 + 2\vec{u}_3 = \underbrace{\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}}_{\substack{3\times 3 \\ A}} \underbrace{\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}}_{\substack{3\times 1 \\ \vec{x}}}$$

**Ex 3:** Write the system of equations
$$\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \qquad - 5x_3 &= -2 \end{aligned}$$
as a

a) Vector Equation

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

property (5) on next page

b) Matrix Equation

$$\begin{bmatrix} 3 & -1 & -4 \\ 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

property (4) on next page

c) Augmented Matrix

$$\begin{bmatrix} 3 & -1 & -4 & 3 \\ 1 & 0 & -5 & -2 \end{bmatrix}$$

property (6) on the next page.

# 1.4: Matrix Equations

**Theorem 3**

If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \qquad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \qquad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \qquad (6)$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solutions if and only if $\mathbf{b}$ is a linear combination of the columns of $A$.

**Ex 4:** Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $b_1, b_2, b_3$? Let's put this in echelon form,

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{array}\right] \begin{array}{l} \\ R_2 + 3R_1 \to R_2 \\ R_3 - 5R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{array}\right] \; R_3 + 2R_2 \to R_3$$

$$\left[\begin{array}{ccc|c} \boxed{1} & -3 & -4 & b_1 \\ 0 & \boxed{-7} & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 + 2b_2 + b_1 \end{array}\right]$$

← the system will be inconsistent if $0 \;\; 0 \;\; 0$ "not zero"

∴ No, the system is only consistent when $b_3 + 2b_2 + b_1 = 0$.

## Theorem 4

Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent. they are all true statements or they are all false.

$\forall$ = "for all"
$\exists$ = "there exists"

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

c. The columns of $A$ span $\mathbb{R}^m$. ← $\forall \vec{v} \in \mathbb{R}^m$ there exist scalars $c_1, \dots, c_N$ s.t. $c_1 a_1 + \dots + c_N a_N = \vec{v}$.

$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N$

d. $A$ has a pivot position in every row.

picture: $\begin{bmatrix} \square & \square & \square \\ & & \square \end{bmatrix}$ w/ $M \leq N$

(Warning: $A$ is a coefficient matrix here, not an augmented matrix.)

**Ex 5:** Compute $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & -3 \\ -3 & -2 & 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & -3 \\ -3 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 4x_2 - x_3 \\ 2x_2 - 3x_3 \\ -3x_3 - 2x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$$

**Row–Vector Rule for Computing $A\mathbf{x}$**

If the product $A\mathbf{x}$ is defined, then the $i$th entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row $i$ of $A$ and from the vector $\mathbf{x}$.

**Ex 6:** Compute

a) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1(1) + (-2)(2) + 3(5) \\ 0(1) + 4(2) + (-1)(5) \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \end{bmatrix}$

b) $\overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \begin{bmatrix} 1(a) + 0(b) + 0(c) \\ 0(a) + 1(b) + 0(c) \\ 0(a) + 0(b) + 1(c) \end{bmatrix} = \overset{\vec{x}}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$

(This is called the identity matrix, denoted by $I$)

If $I_n$ represents $n$x$n$ identity matrix, then $I_n\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$

## Theorem 5

If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $c$ is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

claim: If $A_{m \times n}$, $u \in \mathbb{R}^n$, and $c$ a scalar then $A(c\vec{u}) = c(A\vec{u})$

proof.

Let $A_{m \times n} = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$, $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, and scalar $c$ be given.

$\Rightarrow A(c\vec{u}) = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \left( c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \right)$

$= \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$

$= cu_1\vec{a}_1 + \cdots + cu_n\vec{a}_n$

$= c(u_1a_1 + \cdots + u_na_n)$

$= c \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

$= c(A\vec{u})$

$\therefore A(c\vec{u}) = c(A\vec{u})$.