A THE LANGUAGE OF MATHEMATICS

One of the challenges in learning calculus is growing accustomed to its precise language and terminology, especially in the statements of theorems. In this section, we analyze a few details of logic that are helpful, and indeed essential, in understanding and applying theorems properly.

Many theorems in mathematics involve an **implication**. If A and B are statements, then the implication $A \implies B$ is the assertion that A implies B:

$$A \implies B: \qquad \textit{If } A \textit{ is true, then } B \textit{ is true.}$$

Statement A is called the **hypothesis** (or premise) and statement B the **conclusion** of the implication. Here is an example: *If m and n are even integers, then $m + n$ is an even integer*. This statement may be divided into a hypothesis and conclusion:

$$\underbrace{m \text{ and } n \text{ are even integers}}_{A} \implies \underbrace{m + n \text{ is an even integer}}_{B}$$

In everyday speech, implications are often used in a less precise way. An example is: *If you work hard, then you will succeed.* Furthermore, some statements that do not initially have the form $A \implies B$ may be restated as implications. For example, the statement, "Cats are mammals," can be rephrased as follows:

$$\text{Let } X \text{ be an animal.} \quad \underbrace{X \text{ is a cat}}_{A} \implies \underbrace{X \text{ is a mammal}}_{B}$$

When we say that an implication $A \implies B$ is true, we do not claim that A or B is necessarily true. Rather, we are making the conditional statement that *if* A happens to be true, *then* B is also true. In the above, if X does not happen to be a cat, the implication tells us nothing.

The **negation** of a statement A is the assertion that A is false and is denoted $\neg A$.

Statement A	Negation $\neg A$
X lives in California.	X does not live in California.
$\triangle ABC$ is a right triangle.	$\triangle ABC$ is not a right triangle.

The negation of the negation is the original statement: $\neg(\neg A) = A$. To say that X does *not not live in California* is the same as saying that X *lives in California*.

EXAMPLE 1 State the negation of each statement.

(a) The door is open and the dog is barking.

(b) The door is open or the dog is barking (or both).

Solution

(a) The first statement is true if two conditions are satisfied (door open and dog barking), and it is false if at least one of these conditions is not satisfied. So the negation is

Either the door is not open *OR* the dog is not barking *(or both)*.

(b) The second statement is true if at least one of the conditions (door open or dog barking) is satisfied, and it is false if neither condition is satisfied. So the negation is

The door is not open *AND* the dog is not barking. ■

Contrapositive and Converse

Two important operations are the formation of the contrapositive and the f
converse of a statement. The **contrapositive** of $A \Longrightarrow B$ is the statemen
then A is false":

Keep in mind that when we form the contrapositive, we reverse the order of A and B. The contrapositive of $A \Longrightarrow B$ is NOT $\neg A \Longrightarrow \neg B$.

The contrapositive of $A \Longrightarrow B$ is $\neg B \Longrightarrow \neg A$.

Here are some examples:

Statement	Contrapositive
If X is a cat, then X is a mammal.	If X is not a mammal, then X is not a cat.
If you work hard, then you will succeed.	If you did not succeed, then you did not work hard.
If m and n are both even, then $m+n$ is even.	If $m+n$ is not even, then m and n are not both even.

A key observation is this:

The contrapositive and the original implication are equivale

The fact that $A \Longrightarrow B$ is equivalent to its contrapositive $\neg B \Longrightarrow \neg A$ is a general rule of logic that does not depend on what A and B happen to mean. This rule belongs to the subject of "formal logic," which deals with logical relations between statements without concern for the actual content of these statements.

In other words, if an implication is true, then its contrapositive is automat
vice versa. In essence, an implication and its contrapositive are two way
same thing. For example, the contrapositive, "If X is not a mammal, then
is a roundabout way of saying that cats are mammals.

The **converse** of $A \Longrightarrow B$ is the *reverse* implication $B \Longrightarrow A$:

Implication: $A \Longrightarrow B$	Converse $B \Longrightarrow A$
If A is true, then B is true.	If B is true, then A is true.

The converse plays a very different role than the contrapositive because
NOT equivalent to the original implication. The converse may be true or fa
original implication is true. Here are some examples:

True Statement	Converse	Converse True or
If X is a cat, then X is a mammal.	If X is a mammal, then X is a cat.	False
If m is even, then m^2 is even.	If m^2 is even, then m is even.	True

A counterexample is an example that satisfies the hypothesis but not the conclusion of a statement. If a single counterexample exists, then the statement is false. However, we cannot prove that a statement is true merely by giving an example.

EXAMPLE 2 An Example Where the Converse Is False Show that the c
m and n are even, then $m+n$ is even," is false.

Solution The converse is, "If $m+n$ is even, then m and n are even." To
converse is false, we display a counterexample. Take $m = 1$ and $n = 3$ (or
of odd numbers). The sum is even (since $1 + 3 = 4$) but neither 1 nor 3 is ev
the converse is false.

EXAMPLE 3 An Example Where the Converse Is True State the cont
converse of the Pythagorean Theorem. Are either or both of these true?

Solution Consider a triangle with sides a, b, and c, and let θ be the ang
side of length c, as in Figure 1. The Pythagorean Theorem states that if
$a^2 + b^2 = c^2$. Here are the contrapositive and converse:

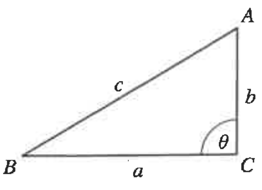


FIGURE 1

Pythagorean Theorem	$\theta = 90^\circ \Longrightarrow a^2 + b^2 = c^2$	True
Contrapositive	$a^2 + b^2 \neq c^2 \Longrightarrow \theta \neq 90^\circ$	Automatically true
Converse	$a^2 + b^2 = c^2 \Longrightarrow \theta = 90^\circ$	True (but not automatic)

The contrapositive is automatically true because it is just another way of stating the orig theorem. The converse is not automatically true since there could conceivably exi nonright triangle that satisfies $a^2 + b^2 = c^2$. However, the converse of the Pythagor Theorem is, in fact, true. This follows from the Law of Cosines (see Exercise 38).

When both a statement $A \Longrightarrow B$ and its converse $B \Longrightarrow A$ are true, we write $A \Leftarrow$ B. In this case, A and B are **equivalent**. We often express this with the phrase

$A \Longleftrightarrow B$ A is true *if and only if* B is true.

For example,

$a^2 + b^2 = c^2$	if and only if	$\theta = 90^\circ$
It is morning	if and only if	the sun is rising.

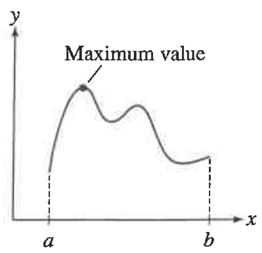
We mention the following variations of terminology involving implications that you n come across:

Statement	Is Another Way of Saying
A is true if B is true.	$B \Longrightarrow A$
A is true only if B is true.	$A \Longrightarrow B$ (A cannot be true unless B is also true.)
For A to be true, it is necessary that B be true.	$A \Longrightarrow B$ (A cannot be true unless B is also true.)
For A to be true, it is sufficient that B be true.	$B \Longrightarrow A$
For A to be true, it is necessary and sufficient that B be true.	$B \Longleftrightarrow A$

Analyzing a Theorem

To see how these rules of logic arise in calculus, consider the following result frc Section 4.2:

THEOREM 1 Existence of Extrema on a Closed Interval A continuous function f o a closed (bounded) interval $I = [a, b]$ takes on both a minimum and a maximum valu on I (Figure 2).



URE 2 A continuous function on a closed erval $I = [a, b]$ has a maximum value.

To analyze this theorem, let's write out the hypotheses and conclusion separately:

Hypotheses A: f is continuous and I is closed.

Conclusion B: f takes on a minimum and a maximum value on I.

A first question to ask is: "Are the hypotheses necessary?" Is the conclusion still true if v drop one or both assumptions? To show that both hypotheses are necessary, we provi counterexamples:

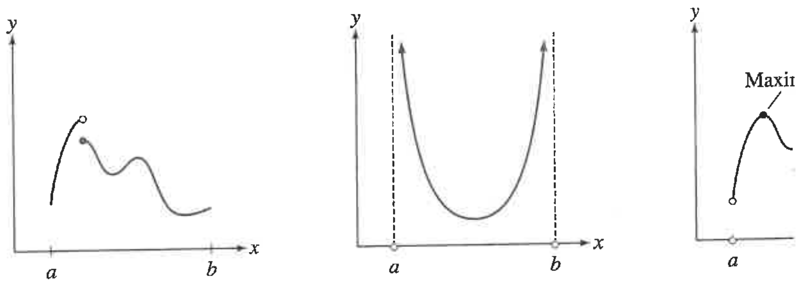
- **The continuity of f is a necessary hypothesis.** Figure 3(A) shows a function on a closed interval $[a, b]$ that is not continuous. This fu maximum value on $[a, b]$, which shows that the conclusion may fail if hypothesis is not satisfied.
- **The hypothesis that I is closed is necessary.** Figure 3(B) shows t continuous function on an *open interval* (a, b). This function has value, which shows that the conclusion may fail if the interval is not

We see that both hypotheses in Theorem 1 are necessary. In stating this, we that the conclusion *always* fails when one or both of the hypotheses are no claim only that the conclusion *may* fail when the hypotheses are not satisfi analyze the contrapositive and converse:

- **Contrapositive $\neg B \implies \neg A$ (automatically true):** If f does not ha and a maximum value on I, then either f is not continuous or I is both).
- **Converse $B \implies A$ (in this case, false):** If f has a minimum and a m on I, then f is continuous and I is closed. We prove this statemer counterexample [Figure 3(C)].

The technique of proof by contradiction *is also known by its Latin name* reductio ad absurdum *or "reduction to the absurd." The ancient Greek mathematicians used proof by contradiction as early as the fifth century* BCE*, and Euclid (325–265* BCE*) employed it in his classic treatise on geometry entitled* The Elements. *A famous example is the proof that $\sqrt{2}$ is irrational in Example 4. The philosopher Plato (427–347* BCE*) wrote: "He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side."*

As we know, the contrapositive is merely a way of restating the theorem, sc ically true. The converse is not automatically true, and in fact, in this case function in Figure 3(C) provides a counterexample to the converse: f ha value on $I = (a, b)$, but f is not continuous and I is not closed.



(A) The interval is closed but the function is not continuous. The function has no maximum value.

(B) The function is continuous but the interval is open. The function has no maximum value.

(C) This functior and the inter but the funct maximum va

FIGURE 3

Mathematicians have devised various general strategies and metho theorems. The method of proof by induction is discussed in Appendix C. An method is **proof by contradiction**, also called **indirect proof**. Suppose our statement A. In a proof by contradiction, we start by assuming that A is false that this leads to a contradiction. Therefore, A must be true (to avoid the c

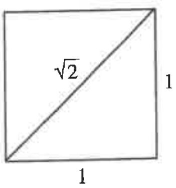


FIGURE 4 The diagonal of the unit square has length $\sqrt{2}$.

■ **EXAMPLE 4 Proof by Contradiction** The number $\sqrt{2}$ is irrational (Fi

Solution Assume that the theorem is false, namely that $\sqrt{2} = p/q$, whe whole numbers. We may assume that p/q is in lowest terms, and therefo of p and q is even. Note that if the square m^2 of a whole number is eve must be even.

The relation $\sqrt{2} = p/q$ implies that $2 = p^2/q^2$ or $p^2 = 2q^2$. This sho be even. But if p is even, then $p = 2m$ for some whole number m, and $p^2 =$ $p^2 = 2q^2$, we obtain $4m^2 = 2q^2$, or $q^2 = 2m^2$. This shows that q is als chose p and q so that at most one of them is even. This contradiction original assumption, that $\sqrt{2} = p/q$, must be false. Therefore, $\sqrt{2}$ is irrat

One of the most famous problems in mathematics is known as "Fermat's Last Theorem." It states that the equation

$$x^n + y^n = z^n$$

has no solutions in positive integers if $n \geq 3$. In a marginal note written around 1630, Fermat claimed to have a proof, and over the centuries, that assertion was verified for many values of the exponent n. However, only in 1994 did the British-American mathematician Andrew Wiles, working at Princeton University, find a complete proof.

CONCEPTUAL INSIGHT The hallmark of mathematics is precision and rigor. A theo is established, not through observation or experimentation, but by a proof that con of a chain of reasoning with no gaps.

This approach to mathematics comes down to us from the ancient Greek ma maticians, especially Euclid, and it remains the standard in contemporary researcl recent decades, the computer has become a powerful tool for mathematical experin tation and data analysis. Researchers may use experimental data to discover pote new mathematical facts, but the title "theorem" is not bestowed until someone wı down a proof.

This insistence on theorems and proofs distinguishes mathematics from the o sciences. In the natural sciences, facts are established through experiment and are sub to change or modification as more knowledge is acquired. In mathematics, theories also developed and expanded, but previous results are not invalidated. The Pythago Theorem was discovered in antiquity and is a cornerstone of plane geometry. In nineteenth century, mathematicians began to study more general types of geometry the type that eventually led to Einstein's four-dimensional space-time geometry in Theory of Relativity). The Pythagorean Theorem does not hold in these more gen geometries, but its status in plane geometry is unchanged.

A. SUMMARY

- The implication $A \Longrightarrow B$ is the assertion, "If A is true, then B is true."
- The *contrapositive* of $A \Longrightarrow B$ is the implication $\neg B \Longrightarrow \neg A$, which says, "If false, then A is false." An implication and its contrapositive are equivalent (one is if and only if the other is true).
- The *converse* of $A \Longrightarrow B$ is $B \Longrightarrow A$. An implication and its converse are not ne sarily equivalent. One may be true and the other false.
- A and B are *equivalent* if $A \Longrightarrow B$ and $B \Longrightarrow A$ are both true.
- In a proof by contradiction (in which the goal is to prove statement A), we star assuming that A is false and show that this assumption leads to a contradiction.

A. EXERCISES

Preliminary Questions

1. Which is the contrapositive of $A \Longrightarrow B$?

(a) $B \Longrightarrow A$ (b) $\neg B \Longrightarrow A$

(c) $\neg B \Longrightarrow \neg A$ (d) $\neg A \Longrightarrow \neg B$

2. Which of the choices in Question 1 is the converse of $A \Longrightarrow B$?

3. Suppose that $A \Longrightarrow B$ is true. Which is then automatically the converse or the contrapositive?

4. Restate as an implication: "A triangle is a polygon."

Exercises

1. Which is the negation of the statement, "The car and the shirt are both blue"?

(a) Neither the car nor the shirt is blue.

(b) The car is not blue and/or the shirt is not blue.

2. Which is the contrapositive of the implication, "If the car has gas, then it will run"?

(a) If the car has no gas, then it will not run.

(b) If the car will not run, then it has no gas.

In Exercises 3–8, state the negation.

3. The time is 4 o'clock.

4. $\triangle ABC$ is an isosceles triangle.

5. m and n are odd integers.

6. Either m is odd or n is odd.

7. x is a real number and y is an integer.

8. f is a linear function.

In Exercises 9–14, state the contrapositive and converse.

9. If m and n are odd integers, then mn is odd.

10. If today is Tuesday, then we are in Belgium.

11. If today is Tuesday, then we are not in Belgium.

12. If $x > 4$, then $x^2 > 16$.

13. If m^2 is divisible by 3, then m is divisible by 3.

14. If $x^2 = 2$, then x is irrational.

In Exercise 15–18, give a counterexample to show that the converse of the statement is false.

15. If m is odd, then $2m + 1$ is also odd.

16. If $\triangle ABC$ is equilateral, then it is an isosceles triangle.

17. If m is divisible by 9 and 4, then m is divisible by 12.

18. If m is odd, then $m^3 - m$ is divisible by 3.

In Exercise 19–22, determine whether the converse of the statement is false.

19. If $x > 4$ and $y > 4$, then $x + y > 8$.

20. If $x > 4$, then $x^2 > 16$.

21. If $|x| > 4$, then $x^2 > 16$.

22. If m and n are even, then mn is even.

In Exercises 23 and 24, state the contrapositive and converse (it is not necessary to know what these statements mean).

23. If f and g are differentiable, then fg is differentiable.

24. If the force field is radial and decreases as the inverse square of the distance, then all closed orbits are ellipses.

In Exercises 25–28, the ***inverse*** *of $A \implies B$ is the implication $\neg A \implies \neg B$.*

25. Which of the following is the inverse of the implication, "If she jumped in the lake, then she got wet"?
(a) If she did not get wet, then she did not jump in the lake.
(b) If she did not jump in the lake, then she did not get wet.
Is the inverse true?

26. State the inverses of these implications:
(a) If X is a mouse, then X is a rodent.
(b) If you sleep late, you will miss class.
(c) If a star revolves around the sun, then it's a planet

27. Explain why the inverse is equivalent to t

28. State the inverse of the Pythagorean Thec

29. Theorem 1 in Section 2.4 states the following: "If tinuous functions, then $f + g$ is continuous." Does it that if f and g are not continuous, then $f + g$ is not c

30. Write out a proof by contradiction for this fact: The positive rational number. Base your proof on the fact th $0 < r/2 < r$.

31. Use proof by contradiction to prove that if $x + y$ or $y > 1$ (or both).

In Exercises 32–35, use proof by contradiction to show is irrational.

32. $\sqrt{\frac{1}{2}}$ 33. $\sqrt{3}$ 34. $\sqrt[3]{2}$

36. An isosceles triangle is a triangle with two equal lowing theorem holds: If Δ is a triangle with two equa is an isosceles triangle.
(a) What is the hypothesis?
(b) Show by providing a counterexample that the hyp sary.
(c) What is the contrapositive?
(d) What is the converse? Is it true?

37. Consider the following theorem: Let f be a quadr with a positive leading coefficient. Then f has a minir
(a) What are the hypotheses?
(b) What is the contrapositive?
(c) What is the converse? Is it true?

Further Insights and Challenges

38. Let a, b, and c be the sides of a triangle and let θ be the angle opposite c. Use the Law of Cosines (Theorem 1 in Section 1.4) to prove the converse of the Pythagorean Theorem.

39. Carry out the details of the following proof by contradiction that $\sqrt{2}$ is irrational (this proof is due to R. Palais). If $\sqrt{2}$ is rational, then $n\sqrt{2}$ is a whole number for some whole number n. Let n be the smallest such whole number and let $m = n\sqrt{2} - n$.
(a) Prove that $m < n$.
(b) Prove that $m\sqrt{2}$ is a whole number.
Explain why (a) and (b) imply that $\sqrt{2}$ is irrational.

40. Generalize the argument of Exercise 39 to prove that $\sqrt{A}$ is irrational if A is a whole number but not a perfect square. *Hint:* Choose n as before and let $m = n\sqrt{A} - n\lfloor\sqrt{A}\rfloor$, where $\lfloor x \rfloor$ is the function.

41. Generalize further and show that for any whole nu root $\sqrt[r]{A}$ is irrational unless A is an rth power. *Hint:* Let that if x is rational, then we may choose a smallest w such that nx^j is a whole number for $j = 1, \dots, r - 1$ $m = nx - n[x]$ as before.

42. Given a finite list of prime numbers p $M = p_1 \cdot p_2 \cdots p_N + 1$. Show that M is not divisibl primes $p_1, \dots, p_N$. Use this and the fact that every nun factorization to prove that there exist infinitely many This argument was advanced by Euclid in *The Elemen*

C INDUCTION AND TH BINOMIAL THEORE

The Principle of Induction is a method of proof that is widely used to pro statement $P(n)$ is valid for all natural numbers $n = 1, 2, 3, \ldots$. Here are of this kind:

- $P(n)$: The sum of the first n odd numbers is equal to n^2.
- $P(n)$: $\frac{d}{dx}x^n = nx^{n-1}$.

The first statement claims that for all natural numbers n,

$$\underbrace{1 + 3 + \cdots + (2n-1)}_{\text{Sum of first } n \text{ odd numbers}} = n^2$$

We can check directly that $P(n)$ is true for the first few values of n:

$P(1)$ is the equality:	$1 = 1^2$	(true)
$P(2)$ is the equality:	$1 + 3 = 2^2$	(true)
$P(3)$ is the equality:	$1 + 3 + 5 = 3^2$	(true)

The Principle of Induction may be used to establish $P(n)$ for all n.

The Principle of Induction applies if $P(n)$ is an assertion defined for $n \geq n_0$, where n_0 is a fixed integer. Assume that

*(i) **Initial step:** $P(n_0)$ is true.*
*(ii) **Induction step:** If $P(n)$ is true for $n = k$, then $P(n)$ is also true for $n = k + 1$.*

Then $P(n)$ is true for all $n \geq n_0$.

THEOREM 1 Principle of Induction Let $P(n)$ be an assertion that c natural number n. Assume that

(i) Initial step: $P(1)$ is true.
(ii) Induction step: If $P(n)$ is true for $n = k$, then $P(n)$ is also true for

Then $P(n)$ is true for all natural numbers $n = 1, 2, 3, \ldots$.

EXAMPLE 1 Prove that $1 + 3 + \cdots + (2n - 1) = n^2$ for all natural n

Solution As above, we let $P(n)$ denote the equality

$$P(n): \quad 1 + 3 + \cdots + (2n - 1) = n^2$$

***Step 1.* Initial step: Show that $P(1)$ is true.**
We checked this above. $P(1)$ is the equality $1 = 1^2$.

***Step 2.* Induction step: Show that if $P(n)$ is true for $n = k$, then $P(n)$ i $n = k + 1$.**
Assume that $P(k)$ is true. Then

$$1 + 3 + \cdots + (2k - 1) = k^2$$

Add $2k + 1$ to both sides:

$$\left[1 + 3 + \cdots + (2k - 1)\right] + (2k + 1) = k^2 + 2k + 1 = (k +$$

$$1 + 3 + \cdots + (2k + 1) = (k + 1)^2$$

This is precisely the statement $P(k + 1)$. Thus, $P(k + 1)$ is true whenev By the Principle of Induction, $P(k)$ is true for all k.

The intuition behind the Principle of Induction is the following. If $P(n)$ were not true for all n, then there would exist a smallest natural number k such that $P(k)$ is false. Furthermore, $k > 1$ since $P(1)$ is true. Thus, $P(k-1)$ is true [otherwise, $P(k)$ would not be the smallest "counterexample"]. On the other hand, if $P(k-1)$ is true, then $P(k)$ is also true by the induction step. This is a contradiction. So $P(k)$ must be true for all k.

EXAMPLE 2 Use Induction and the Product Rule to prove that for all whole numbers n,

$$\frac{d}{dx}x^n = nx^{n-1}$$

Solution Let $P(n)$ be the formula $\frac{d}{dx}x^n = nx^{n-1}$.

Step 1. **Initial step: Show that $P(1)$ is true.**

We use the limit definition to verify $P(1)$:

$$\frac{d}{dx}x = \lim_{h\to 0}\frac{(x+h)-x}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$$

Step 2. **Induction step: Show that if $P(n)$ is true for $n = k$, then $P(n)$ is also true for $n = k+1$.**

To carry out the induction step, assume that $\frac{d}{dx}x^k = kx^{k-1}$, where $k \ge 1$. Then, by the Product Rule,

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x\cdot x^k) = x\frac{d}{dx}x^k + x^k\frac{d}{dx}x = x(kx^{k-1}) + x^k$$
$$= kx^k + x^k = (k+1)x^k$$

This shows that $P(k+1)$ is true.

By the Principle of Induction, $P(n)$ is true for all $n \ge 1$.

In Pascal's Triangle, *the nth row displays the coefficients in the expansion of $(a+b)^n$:*

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 [10 5] 1
6	1 6 15 20 [15] 6 1

The triangle is constructed as follows: Each entry is the sum of the two entries above it in the previous line. For example, the entry 15 in line $n = 6$ is the sum $10 + 5$ of the entries above it in line $n = 5$. The recursion relation guarantees that the entries in the triangle are the binomial coefficients.

As another application of Induction, we prove the Binomial Theorem, which describes the expansion of the binomial $(a+b)^n$. The first few expansions are familiar:

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

In general, we have an expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

where the coefficient of $x^{n-k}x^k$, denoted $\binom{n}{k}$, is called the **binomial coefficient**. Note that the first term in Eq. (2) corresponds to $k = 0$ and the last term to $k = n$; thus $\binom{n}{0} = \binom{n}{n} = 1$. In summation notation,

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$

pansion of $(a+b)^n$. For the inductive step, assume that $P(n)$ is true. By the recu relation, for $1 \le k \le n$, we have

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!}$$
$$= n!\left(\frac{n+1-k}{k!\,(n+1-k)!} + \frac{k}{k!\,(n+1-k)!}\right) = n!\left(\frac{n+1}{k!\,(n+1-k)!}\right)$$
$$= \frac{(n+1)!}{k!\,(n+1-k)!}$$

Thus, $P(n+1)$ is also true and the Binomial Theorem follows by induction.

EXAMPLE 3 Use the Binomial Theorem to expand $(x+y)^5$ and $(x+2)^3$.

Solution The fifth row in Pascal's Triangle yields

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

The third row in Pascal's Triangle yields

$$(x+2)^3 = x^3 + 3x^2(2) + 3x(2)^2 + 2^3 = x^3 + 6x^2 + 12x + 8$$

C. EXERCISES

In Exercises 1–4, use the Principle of Induction to prove the formula for all natural numbers n.

1. $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

2. $1^3+2^3+3^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$

3. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

4. $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$ for any $x \neq 1$

5. Let $P(n)$ be the statement $2^n > n$.

a) Show that $P(1)$ is true.

b) Observe that if $2^n > n$, then $2^n + 2^n > 2n$. Use this to show that if $P(n)$ is true for $n = k$, then $P(n)$ is true for $n = k+1$. Conclude that $P(n)$ is true for all n.

6. Use induction to prove that $n! > 2^n$ for $n \ge 4$.

Let $\{F_n\}$ be the Fibonacci sequence, defined by the recursion formula

$$F_n = F_{n-1} + F_{n-2}, \qquad F_1 = F_2 = 1$$

The first few terms are 1, 1, 2, 3, 5, 8, 13, …. *In Exercises 7–10, use induction to prove the identity.*

7. $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$

8. $F_1^2 + F_2^2 + \cdots + F_n^2 = F_{n+1}F_n$

9. $F_n = \dfrac{R_+^n - R_-^n}{\sqrt{5}}$, where $R_\pm = \dfrac{1 \pm \sqrt{5}}{2}$

10. $F_{n+1}F_{n-1} = F_n^2 + (-1)^n$. *Hint:* For the induction step, show

$$F_{n+2}F_n = F_{n+1}F_n + F_n^2$$
$$F_{n+1}^2 = F_{n+1}F_n + F_{n+1}F_{n-1}$$

11. Use induction to prove that $f(n) = 8^n - 1$ is divisible by 7 fo natural numbers n. *Hint:* For the induction step, show that

$$8^{k+1} - 1 = 7\cdot 8^k + (8^k - 1)$$

12. Use induction to prove that $n^3 - n$ is divisible by 3 for all na numbers n.

13. Use induction to prove that $5^{2n} - 4^n$ is divisible by 7 for all nat numbers n.

14. Use Pascal's Triangle to write out the expansions of $(a+b)^6$ $(a-b)^4$.

15. Expand $(x + x^{-1})^4$.

16. What is the coefficient of x^9 in $(x^3+x)^5$?

17. Let $S(n) = \sum_{k=0}^{n} \binom{n}{k}$.

(a) Use Pascal's Triangle to compute $S(n)$ for $n = 1, 2, 3, 4$.

(b) Prove that $S(n) = 2^n$ for all $n \ge 1$. *Hint:* Expand $(a+b)^n$ evaluate at $a = b = 1$.

18. Let $T(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

(a) Use Pascal's Triangle to compute $T(n)$ for $n = 1, 2, 3, 4$.

(b) Prove that $T(n) = 0$ for all $n \ge 1$. *Hint:* Expand $(a+b)^n$; evaluate at $a = 1, b = -1$.