Math 220: Linear Algebra

Spring 2024 in 17-101, M – F: 10-10:50 am (Class Number #34140)

Spring 2024 in 17-101, M – Th: 11-12:05 pm (Class Number #41439)

Syllabus

Instructor Information

i. Instructor: Dusty Wilson
ii. Office: 26-306
iii. Phone: 206-592-3338
iv. Office Hours:
   - Monday: 9-9:50 am (26-306)
   - Tuesday: 8:30-9:50 am (MRC, 25-6), 12:15-1:30 pm (26-306)
   - Wednesday: none
   - Thursday: 8:30-9:50 am (MRC, 25-6)
   - Friday none
v. home page: http://people.highline.edu/dwilson
vi. e-mail: dwilson@highline.edu

Course Description

Introduction to Linear Algebra: Row operation, matrix algebra; vector spaces, orthogonality, Gram-Schmidt orthogonalization, projections, linear transformations and their matrix representations, rank, similarity; determinants; eigenvalues, eigenvectors, and least squares.

Student Learning Outcomes

i. Solve systems using Gauss-Jordan elimination.
ii. Identify and orthogonalize the basis of a vector space.
iii. Apply matrix methods to model a data set using least squares regression.
iv. Calculate and interpret the eigenvalues and eigenvectors of a matrix.
v. Identify, create, and apply linear transformations using matrix methods.
vi. Construct a mathematical proof.

Text

Linear Algebra and its Applications, 6th ed. by Lay, Lay, and McDonald.

Prerequisite

Math 152 with a minimum grade of 2.0.

Calculators

A graphing calculator is required for this course.

i. The TI-84 calculator is recommended. The use of symbolic calculators may not be allowed during assessments/exams. Furthermore, the use of all calculators may be prohibited during some assessments/exams (forewarning will be given).
ii. Very limited class time will be spent explaining the use of calculators.
iii. Calculators may be rented from the Math Department through the Library.

Canvas

i. Nothing is submitted through Canvas
ii. Reminders of when things are due are available through Canvas (except HW/Quizzes which are only listed in MyLabs or on the course calendar).
iii. Grades are available in Canvas.

Homework and Quizzes

Homework and Quizzes will be assigned on MyLabs online. It is important that you use the online resources to learn the material, and not just "get problems right". Keep an organized notebook of your work, clearly labeling the section and problem number, etc. so you can review your work as it will be helpful on Assessments and in studying for the Final Exam.

i. You will have five attempts on each/most homework questions.
ii. Quizzes are timed and short. The questions parallel the homework assignments. You may attempt each quiz three times, but are allowed only one attempt at each question.
   - If you get quiz questions right, the homework is sometimes personalized so that you don't need to do that part of the homework.
   - You don't have multiple attempts on quiz questions.

Projects

i. There will be four projects assigned during the quarter. The projects are designed to give you a better grasp of the graphs and pictures of linear algebra.
ii. Some will be individual efforts and others may be worked as a small group.
iii. If you miss a project, a score of 0% will be assigned.

Slack Assignments

i. We will use Slack to communicate, share work, coordinate schedules, and for reminders. Slack is sort of like Discord or GroupMe, but is often used in professional settings which is why I've chosen to use it. Other classes use discussion boards, we are using Slack.
ii. Slack: There are four channels we will use on Slack.
   - General: This is where I will make announcements and you should ask general questions about the class (i.e., What sections are on the Assessment?).
   - Homework: This is where you will post questions and answers to homework questions.
   - Study groups: This is where you can post about upcoming study groups.
   - Random: You can post about club meetings, off campus gatherings, and fun stuff.
iii. At least once each week you will make a post on Slack regarding homework. This can be a homework question, homework solution, or a response to a classmate's question.
   - These are graded for completion.
   - There is a 5% bonus for asking a question and a 10% bonus if you respond to a classmate's question.
   - Take a screenshot of your response (including your name and date). Save it as a pdf, and upload this into Gradescope.
   - You will be posting homework questions/responses. This is also where I will make announcements and can best be asked questions outside of class.
iv. Slack assignments will be submitted in Gradescope.
   - One assignment score will be dropped.

Discussion Seminars

i. There will be an opportunity to read an article or watch a video related to math, philosophy, history, studenting, etc. You will respond in writing to provided prompts and then discuss the article/video and your writing with your peers.

ii. Half the credit in this category will be for preparation for the seminar (measured through the electronic submission) and the other half for attendance (measured through bringing printed/typed notes to class).

iii. One assignment score will be dropped (but not the Letter to a Future Student).

iv. Discussion Seminar 0 is an outlier and simply exists to help make sure you can interact with the various (and sometimes confusing) technologies we will use in this class.

Notes/Slack/Discussions submitted via Gradescope

i. Handwritten work will be submitted (and returned) through a free program called Gradescope. It is similar to uploading into Canvas, but is designed for math/science classes and so easier for me to administer.

ii. Once each week you will scan your class notes and upload the pdf into Gradescope.
- These are graded for completion.
- One score will be dropped.

iii. Weekly Slack Assignments (explained previously) are submitted via Gradescope.

iv. Discussion Seminar notes (explained previously) are submitted via Gradescope. These are due before class to encourage you to be prepared for the discussion and also to provide an electronic backup should you forget to bring the printed copy.

Assessments

There will be (near) weekly assessments:

i. The assessments will be cumulative, but will emphasize recent material.

ii. The length/value of the assessments will vary but will typically be about 30 minutes.

iii. Your lowest assessment score will be dropped if you attend 60%+ of the class days where attendance is taken.

iv. All assessments must be taken during the scheduled class time.
- Other arrangements can be made under special circumstances.

v. Spoken and written communication as well as sharing of calculators during exams is prohibited.

Final Exam

A comprehensive final exam will be held in the regular class meeting room. See the quarterly class schedule for dates and times. The final exam is mandatory and a grade of 0.0 may be assigned at the instructor's discretion to those who fail to take the final exam.

Errors

I am human and fallible. If you find possible typos or math errors in printed material, videos, or that were made in class, please let me know. If these are in print materials or video, please DM me in Slack and make sure to include a screen shot and link so that I can easily find the mistake and fix it. Sometimes there is extra credit when errors are pointed out. I'm also a sensitive soul … so please be considerate when pointing out possible errors.

Working remotely and illnesses

It is a brave new world where we face new challenges such as Covid-19 while also having many new tools and skills in the areas of teaching and learning:

i. Attending class in person is a good thing.

ii. Lessons will generally be available online:
- For when you are sick and unable to attend class.
- For when you want to review a lesson.
- For when you can't (or don't want to) make it to class.
- For when face-to-face class is cancelled because I am sick or otherwise unable to teach.

iii. Assessments, exams, and discussion seminars must be attended face-to-face. Plan accordingly. These cannot be completed remotely.[1]

iv. Because assessments must be completed in person, one score will be dropped if your attendance score is at least 60%. Special circumstances can be addressed on a case-by-case basis.

[1] Perhaps it goes without saying, that remote options would be available should on campus classes be cancelled. Similarly, please talk to me if you have a prolonged illness (or the like).

Grading

Homework: 4%
Online quizzes: 4%
Attendance: 2%
Slack Assignments: 2%
Discussion Seminars: 2%
Class notes: 2%
Project: 4%
Assessments: 50%
Final Exam: 30%.

GPA's will be given according to:

95-100%	4.0	%%	GPA	%%%	GPA	%%	GPA	%%	GPA
93-4%	3.9	81%	3.1	73%	2.3	65%	1.5	57%	0.7
91-2%	3.8	80%	3.0	72%	2.2	64%	1.4	0-56%	0.0
89-90%	3.7	79%	2.9	71%	2.1	63%	1.3		
87-8%	3.6	78%	2.8	70%	2.0	62%	1.2		
85-6%	3.5	77%	2.7	69%	1.9	61%	1.1		
84%	3.4	76%	2.6	68%	1.8	60%	1.0		
83%	3.3	75%	2.5	67%	1.7	59%	0.9		
82%	3.2	74%	2.4	66%	1.6	58%	0.8		

Policies and Notes

i. **Attendance**: You are responsible for all material covered in class including all announced changes to the schedule and assigned course work. (If you miss class, *you* are still responsible for everything in class).

ii. **Devices**: The use of non-human smart gadgets in class is discouraged (except when requested). Smart non-human devices are banned during assessments and tests.

iii. **Math Resource Center**: Cost-free mathematics tutoring is available at the MRC. The MRC is located on the sixth floor of the library (Bldg 25).

iv. **Faculty Advising**: Highline College instructors are a wonderful resource for students at any stage of the academic process. Many Highline instructors have career experience, are knowledgeable about campus resources, and can assist students in reaching their educational goals through degree planning. If you have an advising question, feel free to approach your instructor. If your instructor cannot answer your question, s/he will help you find someone who can.

v. **Honors**: Highline College offers opportunities for students to participate in an Honors Program tailored to their pathways. Students who fulfill all Honors Program requirements may become eligible for a scholarship during their final quarter and receive recognition at Highline's commencement ceremony.
If you are interested in the Honors Program, I invite you to pursue an honors project in this class. Please approach me within the first three weeks of the quarter, and we will work together to develop a plan for completing an advanced academic or professional project. After completing the project and earning a 3.5 GPA in this course, an "honors" notation will appear on your official Highline transcript.

vi. **Academic Dishonesty**: Cheating, plagiarism, and other forms of academic dishonesty are unacceptable at Highline College and may result in lower grades and/or disciplinary action. It is both your right and responsibility to be familiar with the document entitled: Student Rights and Responsibilities code WAC 132I-1210 adopted by the Board of Trustees of Community College District 9 on December 13, 2007. This is available in the counseling center.

vii. **Access Services**: Your experience in this class is important to me. If you have already established accommodations with Access Services, please communicate your approved accommodations to me at your earliest convenience so we can discuss your needs in this course. If you have not yet established services through Access Services but have a temporary health condition or permanent disability that requires accommodations (conditions include but are not limited to; mental health, attention-related, learning, vision, hearing, physical or health impacts), you are welcome to contact Access Services at 206-592-3857, access@highline.edu or access.highline.edu. Access Services is located on the 5th floor of the Library, Building 25, Room 531.

viii. **Emergency Procedures**: In the event of an emergency, follow your instructor's directions. If you are told to evacuate the building, take your valuables because you may not be allowed to re-enter. Do not leave campus until your instructor or another campus official tells you to do so. If you may need assistance evacuating, notify your instructor today. To prepare yourself for an emergency, review the evacuation map on the last page of the emergency placard in your classroom and subscribe to HC Alert at https://hctextalerts.highline.edu/).

ix. **Final Exams**: Your completed final exam will not be returned to you. It belongs to the instructor. However, you may (and should) review your final exam by stopping by the instructor's office the next quarter.

x. **School Policies**:

   i. The Student Rights and Responsibilities Code: A legal document that describes college expectations, students' rights, and outlines the process for resolving disciplinary matters and Code violations. http://studentservices.highline.edu/srr.php

   ii. The College Catalog: Lots of fine print about grades, deadlines, and resources can be found in the catalog at: http://catalog.highline.edu/

x. **Important Dates (dates should be verified online)**:

   i. April 5th: Last Day for 100% Tuition Refund

   ii. April 12th: The last day to drop without incurring a "W"

   iii. May 24th: The last day to officially withdraw with a "W"

Pearson | MyLab | Math

Student Registration Instructions

To register for Math 220: Linear Algebra:

1. Go to https://mlm.pearson.com/enrollment/wilson79457
2. Sign in with your Pearson student account or create your account.

   For Instructors creating a Student account, do not use your instructor credentials.
3. Select any available access option, if asked.
   - » Enter a prepaid access code that came with your textbook or from the bookstore.
   - » Buy instant access using a credit card or PayPal.
   - » Select **Get temporary access without payment**.
4. Select **Go to my course**.
5. Select **Math 220: Linear Algebra** from My Courses.

If you contact Pearson Support, give them the course ID: wilson79457

To sign in later:

1. Go to https://mlm.pearson.com
2. Sign in with the same Pearson account you used before.
3. Select **Math 220: Linear Algebra** from My Courses.

Date		Topic	MyLabs	Gradescope
5/9	Thu	**Assessment 4 (4.1-2), Discussion Seminar IV**		
5/10	Fri	4.5: Dimension and Rank		
5/11	Sat	Weekend - No Class		
5/12	Sun	Weekend - No Class	4.4 HW & Q	
5/13	Mon	4.6: Change of Basis		Slack 6 and Notes 6 (4.2-4)
5/14	Tue	5.1: Eigenvectors & Eigenvalues	4.5 HW & Q	
5/15	Wed	Review	4.6 HW & Q	
5/16	Thu	**Assessment 5 (4.3-5), Discussion Seminar V**		
5/17	Fri	5.2: The Characteristic Equation		Project 2 due
5/18	Sat	Weekend - No Class		
5/19	Sun	Weekend - No Class	5.1 HW & Q	Slack 7 and Notes 7 (4.5-4.6)
5/20	Mon	5.3: Diagonalization		
5/21	Tue	5.4-6: Dynamical Systems	5.2 HW & Q	
5/22	Wed	Review	5.3 HW & Q	
5/23	Thu	**Assessment 6 (5.1-3), Discussion Seminar VI**		
5/24	Fri	5.4-6: Dynamical Systems		Project 3 due (in person)
5/25	Sat	Weekend - No Class		
5/26	Sun	Weekend - No Class		Slack 8 and Notes 8 (5.1-3)
5/27	Mon	Memorial Day - No School		
5/28	Tue	6.1: Inner Product, Length, & Orthogonality		
5/29	Wed	6.2: Orthogonal Sets	5.4-6 HW & Q	
5/30	Thu	6.3 & 6.4: Orthogonal Projections & Gram-Schmidt	6.1 HW & Q	
5/31	Fri	6.3 & 6.4: Orthogonal Projections & Gram-Schmidt		
6/1	Sat	Weekend - No Class		
6/2	Sun	Weekend - No Class	6.2 HW & Q	Slack 9 and Notes 9 (5.4-6.1)
6/3	Mon	6.5 & 6.6: Least-Squares Problems		Project 4 due in person
6/4	Tue	**Assessment 7 (5.4-6, 6.1-2), Discussion Seminar VII**		
6/5	Wed	Review		
6/6	Thu	Review		
6/7	Fri	Review		
6/8	Sat	Weekend - No Class		
6/9	Sun	Weekend - No Class	6.3/4 HW & Q	Slack 10 and Notes 10 (6.2-6.6)
6/10	Mon	**Final Exam (10 - 11:50 am)**	6.5/6 HW & Q	
6/11	Tue	**Final Exam (11 - 12:50 pm)**		
6/12	Wed			
6/13	Thu	**Commencement at the Showare Center**		

Calendar - 10am

Date		Topic	MyLabs	Gradescope
4/1	Mon	Introductions, Thinking 1		
4/2	Tue	Thinking 2		Discussion Seminar 0
4/3	Wed	Thinking 3		
4/4	Thu	1.1: Systems of Linear Equations		
4/5	Fri	1.2: Row Reduction and Echelon Form		
4/6	Sat	Weekend - No Class		
4/7	Sun	Weekend - No Class		Slack 1 and Notes 1 (1.1)
4/8	Mon	1.3: Vector Equations		
4/9	Tue	1.4: The Matrix Equation Ax=b	1.1 HW & Q	
4/10	Wed	Review	1.2 HW & Q	<- HW = MyLabs homework
4/11	Thu	**Assessment 1 (1.1-2), Discussion Seminar I**		and Q = MyLabs quiz
4/12	Fri	1.5: Solution Sets of Linear Systems		
4/13	Sat	Weekend - No Class		
4/14	Sun	Weekend - No Class	1.3 HW & Q	Slack 2 and Notes 2 (1.2-1.3)
4/15	Mon	1.7: Linear Independence		
4/16	Tue	1.8: Linear Transformations	1.4 HW & Q	
4/17	Wed	1.9: Matrix of a Linear Transformation	1.5 HW & Q	
4/18	Thu	Review		
4/19	Fri	Math Conference - No Class		Project 1 due
4/20	Sat	Weekend - No Class		
4/21	Sun	Weekend - No Class	1.6&7 HW & Q	Slack 3 and Notes 3 (1.4-1.7)
4/22	Mon	2.1: Matrix Operations	1.8 HW & Q	
4/23	Tue	2.2: Inverse of a Matrix	1.9 HW & Q	
4/24	Wed	Review	2.1 HW & Q	
4/25	Thu	**Assessment 2 (1.3-9), Discussion Seminar II**		
4/26	Fri	2.3: Characteristics of Invertible Matrices		
4/27	Sat	Weekend - No Class		
4/28	Sun	Weekend - No Class	2.2 HW & Q	Slack 4 and Notes 4 (1.8-2.2)
4/29	Mon	3.1 & 3.2: Determinants		
4/30	Tue	4.1: Vector Spaces and Subspaces	2.3 Q	
5/1	Wed	4.2: Null Spaces, Column Spaces, and Linear Transformations	3.1-2 HW & Q	
5/2	Thu	**Assessment 3 (2.1-3), Discussion Seminar III**		
5/3	Fri	Equity Day - All classes cancelled		
5/4	Sat	Weekend - No Class		
5/5	Sun	Weekend - No Class	4.1 HW & Q	Slack 5 and Notes 5 (2.3-4.1)
5/6	Mon	4.3: Linearly Independent Sets; Bases		
5/7	Tue	4.4: Coordinates	4.2 HW & Q	
5/8	Wed	Review	4.3 HW & Q	

Tentative and subject to change

Date	Topic	MyLabs	Gradescope
5/9 Thu	**Assessment 4 (4.1-2), Discussion Seminar IV**		
5/10 Fri	No class on Fridays		
5/11 Sat	Weekend - No Class		
5/12 Sun	Weekend - No Class	4.4 HW & Q	
5/13 Mon	4.6: Change of Basis		Slack 6 and Notes 6 (4.2-4)
5/14 Tue	5.1: Eigenvectors & Eigenvalues	4.5 HW & Q	
5/15 Wed	5.2: The Characteristic Equation	4.6 HW & Q	
5/16 Thu	**Assessment 5 (4.3-5), Discussion Seminar V**		
5/17 Fri	No class on Fridays		Project 2 due
5/18 Sat	Weekend - No Class		
5/19 Sun	Weekend - No Class	5.1 HW & Q	Slack 7 and Notes 7 (4.5-4.6)
5/20 Mon	5.3: Diagonalization		
5/21 Tue	5.4-6: Dynamical Systems	5.2 HW & Q	
5/22 Wed	5.4-6: Dynamical Systems	5.3 HW & Q	
5/23 Thu	**Assessment 6 (5.1-3), Discussion Seminar VI**		
5/24 Fri	No class on Fridays		Project 3 due (in person)
5/25 Sat	Weekend - No Class		
5/26 Sun	Weekend - No Class		Slack 8 and Notes 8 (5.1-3)
5/27 Mon	Memorial Day - No School		
5/28 Tue	6.1: Inner Product, Length, & Orthogonality		
5/29 Wed	6.2: Orthogonal Sets	5.4-6 HW & Q	
5/30 Thu	6.3 & 6.4: Orthogonal Projections & Gram-Schmidt	6.1 HW & Q	
5/31 Fri	No class on Fridays		
6/1 Sat	Weekend - No Class		
6/2 Sun	Weekend - No Class	6.2 HW & Q	Slack 9 and Notes 9 (5.4-6.1)
6/3 Mon	6.5 & 6.6: Least-Squares Problems		Project 4 due in person
6/4 Tue	**Assessment 7 (5.4-6, 6.1-2), Discussion Seminar VII**		
6/5 Wed	Review		
6/6 Thu	Review		
6/7 Fri	No class on Fridays		
6/8 Sat	Weekend - No Class		
6/9 Sun	Weekend - No Class	6.3/4 HW & Q	Slack 10 and Notes 10 (6.2-6.6)
6/10 Mon	**Final Exam (10 - 11:50 am)**	6.5/6 HW & Q	
6/11 Tue	**Final Exam (11 - 12:50 pm)**		
6/12 Wed			
6/13 Thu	**Commencement at the Showare Center**		

Calendar - 11am

Date		Topic	MyLabs	Gradescope
4/1	Mon	Introductions, Thinking 1		
4/2	Tue	Thinking 2		Discussion Seminar 0
4/3	Wed	Thinking 3		
4/4	Thu	1.1: Systems of Linear Equations		
4/5	Fri	No class on Fridays		
4/6	Sat	Weekend - No Class		
4/7	Sun	Weekend - No Class		Slack 1 and Notes 1 (1.1)
4/8	Mon	1.2: Row Reduction and Echelon Form		
4/9	Tue	1.3: Vector Equations	1.1 HW & Q	
4/10	Wed	1.4: The Matrix Equation Ax=b	1.2 HW & Q	<- HW = MyLabs homework
4/11	Thu	**Assessment 1 (1.1-2), Discussion Seminar I**		and Q = MyLabs quiz
4/12	Fri	No class on Fridays		
4/13	Sat	Weekend - No Class		
4/14	Sun	Weekend - No Class	1.3 HW & Q	Slack 2 and Notes 2 (1.2-1.3)
4/15	Mon	1.5: Solution Sets of Linear Systems		
4/16	Tue	1.7: Linear Independence	1.4 HW & Q	
4/17	Wed	1.8: Linear Transformations	1.5 HW & Q	
4/18	Thu	1.9: Matrix of a Linear Transformation		
4/19	Fri	No class on Fridays		Project 1 due
4/20	Sat	Weekend - No Class		
4/21	Sun	Weekend - No Class	1.6&7 HW & Q	Slack 3 and Notes 3 (1.4-1.7)
4/22	Mon	2.1: Matrix Operations	1.8 HW & Q	
4/23	Tue	2.2: Inverse of a Matrix	1.9 HW & Q	
4/24	Wed	2.3: Characteristics of Invertible Matrices	2.1 HW & Q	
4/25	Thu	**Assessment 2 (1.3-9), Discussion Seminar II**		
4/26	Fri	No class on Fridays		
4/27	Sat	Weekend - No Class		
4/28	Sun	Weekend - No Class	2.2 HW & Q	Slack 4 and Notes 4 (1.8-2.2)
4/29	Mon	3.1 & 3.2: Determinants		
4/30	Tue	4.1: Vector Spaces and Subspaces	2.3 Q	
5/1	Wed	4.2: Null Spaces, Column Spaces, and Linear Transformations	3.1-2 HW & Q	
5/2	Thu	**Assessment 3 (2.1-3), Discussion Seminar III**		
5/3	Fri	Equity Day - All classes cancelled		
5/4	Sat	Weekend - No Class		
5/5	Sun	Weekend - No Class	4.1 HW & Q	Slack 5 and Notes 5 (2.3-4.1)
5/6	Mon	4.3: Linearly Independent Sets; Bases		
5/7	Tue	4.4: Coordinates	4.2 HW & Q	
5/8	Wed	4.5: Dimension and Rank	4.3 HW & Q	

Tentative and subject to change

LINEAR ALGEBRA PROJECT
Part 1: Span

Instructions: You will be given two vectors in for R^2 from which to make a new parallelogram grid (on top of an ordinary Cartesian grid). You will also be given two vectors to locate on the two grids.

0. Put your name and project number at the top of each page. Your work throughout should be neat ... very neat and organized. Work on graph paper.

1. Graph $\vec{v}_1$ in red and $\vec{v}_2$ in blue.

2. Create a parallelogram grid using $\vec{v}_1$ and $\vec{v}_2$. (This is a foreshadow of *B*-coordinates).

3. Treating the vector $\vec{x}$ like a position vector, how many $\vec{v}_1$'s and $\vec{v}_2$'s are required to get to $\vec{x}$. (This is a foreshadow of finding $[\vec{x}]_B$ given $\vec{x}$).

4. Graph the vector that is _____ units in the $\vec{v}_1$ direction and _____ units in the $\vec{v}_2$ direction. Label this point as $\vec{y}$ and find its coordinates on the Cartesian grid. (This is a foreshadow of finding $\vec{y}$ given $[\vec{y}]_B$).

Submit your graphs and work via Gradescope

LINEAR ALGEBRA PROJECT
Part 2: Change of Bases

Instructions: You will be given a new basis for R^2, a vector $\vec{x}$, and a specific transformation.

0. Put your name and project number at the top of each page. Your work throughout should be neat … very neat and organized.

1. Find the matrix A for the transformation.

2. Find $T(\vec{x})$ for this transformation.

3. Find $[\vec{x}]_B$

4. Find the transformation B.

5. Using A, B, and P (aka P_B), show that A and B are similar. To do this, you need to actually multiply finding two products and showing that they are equal.

6. Using graph paper and careful scaling, create an overlay "graph paper" with the basis vectors from your new basis. Label your basis vectors and show $\vec{v}_1$ in red and $\vec{v}_2$ in blue.

7. Graph $[\vec{x}]_B$ and $[T(\vec{x})]_B$ on your new graph paper.

8. Clearly and carefully label $[\vec{x}]_B$ and $[T(\vec{x})]_B$ with coordinates relative to the new basis, and relative to the standard basis.

If you finish this early, you can check it with me prior to submission.

Submit your graphs and work via Gradescope.

LINEAR ALGEBRA PROJECT
Part 3: Four Major Subspaces

Instructions:

1. Read carefully through Part 3 and 4 of the project.

2. The math on paper
   a. Create a non-trivial 3x3 matrix A that is not invertible (this should not match an example in the book). A should have two linearly independent rows.
   b. Find bases for:
      i. The row space of A (you may need to look up the row space)
      ii. The image of A
      iii. The kernel of A, and
      iv. The kernel of A^T (you may need to look up the transpose)
   c. Check with me to make sure your A and these bases are correct. It is nice to know the math is right before you begin to build.

3. Model 1: Row space and kernel
   a. Put your name(s) on your models
   b. Build a 3D model using your ingenuity and creativity that shows the row space, kernel, and the relationship between the subspaces.
   c. This should be built to scale with labeled axes (label x, y, z and also the scale 1, 2, 3, …)
   d. Make sure all vectors *and* subspaces are clearly labeled.
   e. Note: Should you find it helpful, it is acceptable to adjust your matrix A to allow for easier modeling.
   f. Hint: You may find it helpful to look at the grading rubric to see what I am looking for.

4. Model 2: Image and the transpose of the kernel
   a. Put your name(s) on your models
   b. Build a 3D model using your ingenuity and creativity that shows the image, transpose of the kernel, and the relationship between the subspaces.
   c. This should be built to scale with labeled axes (label x, y, z and also the scale 1, 2, 3, …)
   d. Make sure all vectors *and* subspaces are clearly labeled.

LINEAR ALGEBRA PROJECT
Part 3: Four Major Subspaces

Rubric:

1. Create a matrix A that is not invertible (this should not match an example in the book) size 3x3.
   a. 0 points

2. Find bases for the row space of A, image of A, kernel of A, and kernel of A^T
   a. 1 point for each subspace.
   b. -0.5 if a redundant vector
   c. -0.5 if span is used incorrectly

3. Suggestion: Check your A and subspaces with me prior to beginning to build.

4. By the deadline have two displays (3D models), one with row space and kernel of A indicated, and another with the image of A and the kernel of A^T indicated.
   a. Display 1: row space and kernel of A
      _____ Correct vectors
      _____ Orthogonal
      _____ Span

   b. Display 2: image of A and the kernel of A^T
      _____ Correct vectors
      _____ Orthogonal
      _____ Span

5. Late submissions will be accepted within reason, but will have 2 points deducted from their final score.

Total points = 10

LINEAR ALGEBRA PROJECT
Part 4: Projections and Orthogonal Bases

Instructions: This project asks that you add to your previously constructed 3D models.

1. On Model 1 (row space and kernel):
   a. Choose a vector $\vec{x}$ that does not lie in either the row space or the image.
      i. Hint: A careful choice of $\vec{x}$ can make the subsequent projections easy to find/graph.
   b. Find the projection of this vector onto the basis vectors for the row space and find the projection of this vector onto the basis vector for ker(A).
   c. Add the vector and its projections to Model 1.
      i. Label all four vectors!

2. On Model 2 (image and kernel of the transpose):
   a. Create an orthonormal basis for your image of A.
   b. Add these two vectors to Model 2.
      i. Check your work making sure to label all vectors!

3. On Model 2 (image and kernel of the transpose):
   a. Using the same basis you used last time for the row space of A create a vector, $\vec{v}$, such that $\vec{v}$ is the linear combination of the basis vectors: $\vec{v} = \vec{b}_1 + 2\vec{b}_2$. Let $T : \Re^3 \rightarrow \Re^3$ be such that $T(\vec{v}) = A\vec{v}$
      i. Find $T(\vec{b}_1)$, $T(\vec{b}_2)$, and $T(\vec{v})$
      ii. Write $T(\vec{v})$ as a linear combination of $T(\vec{b}_1)$ and $T(\vec{b}_2)$
      iii. Put all three vectors from part (ii.) on the model of your image. If they don't fit on your model, scale them down and label them accordingly.
         1. In any case, label the vectors.

1.1: Systems of Linear Equations

Math 220: Linear Algebra

Linear Equations

A ________________ ________________ of the variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Where b, a_1, a_2, ..., a_n are real or complex numbers.

Ex 1: Circle the linear equations, and state why the non-linear aren't linear.

$a)\ 3x_1 - x_2 = 5x_1$

$b)\ 4x_2 + 5 = \sqrt{x_1}$

$c)\ x_1 - 4x_2 = x_1x_2$

$d)\ -2x_1 + 7x_2 - \pi x_3 = \sqrt{2}$

A ________________ ____ ________________ ________________ is a collection of one or more linear equations with the same variables. For example

$$\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \qquad - 5x_3 &= -2 \end{aligned}$$

A ________________ of a system is a list of numbers $(s_1, s_2, \ldots, s_n)$ that make every equation of the system true, when each s_k is substituted for x_k.

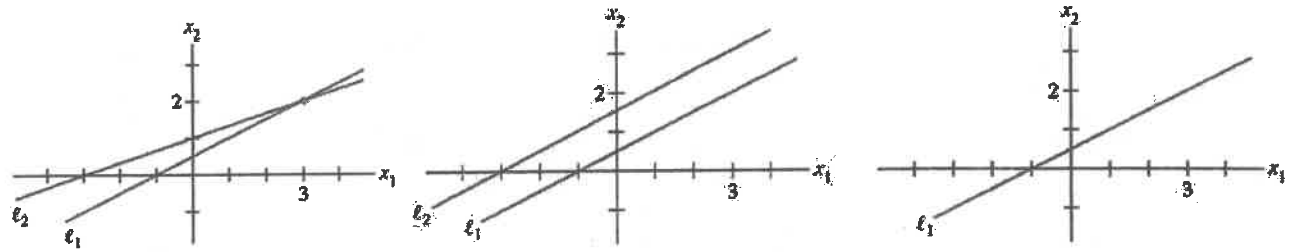
Ex 2: Verify that (3,2,1) is a solution to the system $\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \qquad - 5x_3 &= -2 \end{aligned}$.

The set of all possible solutions is called the ________________ ________.

Ex 3: Find another solution to the system from Ex 2.

Two systems are considered ______________________ if they have the same solution set.

From 2 dimensional systems of equations in algebra, we should remember that there are 3 possibilities for the number of solutions to a system.



A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system is called ______________________ if it has at least one solution, and ______________________ if it has no solutions.

Matrix Notation

We will represent a system of equations by its coefficients in a ________________.

$$\begin{aligned} x_1 \qquad\quad -3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

will be re-written as the

________________ matrix

The __________ of a matrix tells how many __________ and ____________ a matrix has.

________________ matrix

An $m \times n$ matrix has

___ ____________ and

___ ________________.

1.1: Systems of Linear Equations

Solving a Linear System – We are going to describe an algorithm for solving linear systems, which replaces one system with an equivalent one that is easier to solve. Since they are equivalent, they have the same solution set.

Ex 4: Solve the system

$$\begin{aligned} x_1 \qquad -3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

Three Operations we can use:

1.1: Systems of Linear Equations

Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Two matrices are called __________ ______________________ if there are a sequence of elementary row operations that transform one matrix into the other.

If two systems are row equivalent, they have the same __________________ _____.

Two Fundamental Questions About a Linear System

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

Ex 5: Determine whether the systems are consistent or inconsistent. Do not fully solve.

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

Ex 6: Determine whether the systems are consistent or inconsistent. Do not fully solve.

$$\begin{aligned} x_1 \quad\quad + 3x_3 \quad\quad &= 2 \\ x_2 \quad\quad - 3x_4 &= 3 \\ -2x_2 + 3x_3 + 2x_4 &= 1 \\ 3x_1 \quad\quad\quad\quad + 7x_4 &= -5 \end{aligned}$$

1.2: Row Reduction & Echelon Forms

Math 220: Linear Algebra

Definition

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

The following matrices that we saw in section 1.1 are in

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

________________ ________________

Ex 1: Here are matrices in

Echelon Form

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Reduced Echelon Form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Nonzero matrices can be row-reduced into many different matrices in Echelon form. However, the Reduced Echelon Form of any matrix is unique – there is only one.

Theorem 1 Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

Definition

A **pivot position** in a matrix *A* is a location in *A* that corresponds to a leading 1 in the reduced echelon form of *A*. A **pivot column** is a column of *A* that contains a pivot position.

Ex 2: Row reduce the matrix to echelon form, and locate the pivot columns.

$$\begin{bmatrix} -3 & 1 & -18 & -5 & 4 & 4 \\ 1 & 1 & 2 & 3 & 1 & 1 \\ -1 & 1 & -8 & -1 & 0 & 0 \\ 1 & 2 & -1 & 4 & -5 & -5 \end{bmatrix}$$

1.2: Row Reduction & Echelon Forms

Ex 3: Use elementary row operations to transform the following matrix into echelon form and then reduced echelon form.

$$\begin{bmatrix} 2 & -4 & 3 & -4 & -11 & 28 \\ -1 & 2 & -1 & 2 & 5 & -13 \\ 0 & 0 & -3 & 1 & 6 & -10 \\ 3 & -6 & 10 & -8 & -28 & 61 \end{bmatrix}$$

Forward Phase vs. Backward Phase

Solutions of Linear Systems

Ex 4: (revisited) Looking at the reduced echelon form of the matrix from Ex 3, we can describe our solution set to the corresponding system of equations to this augmented matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

The variables that are arbitrary, this text calls ________ variables, and the others that rely on those ________ variables or are fixed are called ____________ variables.

Ex 5: Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 0 & -2 & 4 & 3 & 3 \\ 0 & 1 & 3 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Theorem 2 Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \cdots \ 0 \ b] \quad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Ex 6: Determine the existence and uniqueness of the linear systems represented by the augmented matrices that we've seen over the last two sections.

a) (1.1, #4)

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

b) (1.1, #5)

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

c) (1.2, Ex 3 revisited)

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

4. Write the system of equations corresponding to the matrix obtained in step 3.

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.3: Vector Equations

Math 220: Linear Algebra

Vectors in $\mathbb{R}^2$

A matrix with one column is called a ________________ ____________ or ____________

$$\mathbf{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} \\ \\ \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Vectors are __________________ if and only if the corresponding entries are equal.

The sum of the vectors $\mathbf{u}$ and $\mathbf{v}$ is the vector __________________.

The scalar multiple of vector $\mathbf{w}$ by a real number c is the vector $c\mathbf{w}$ where each ____________ of $\mathbf{w}$ is multiplied by c.

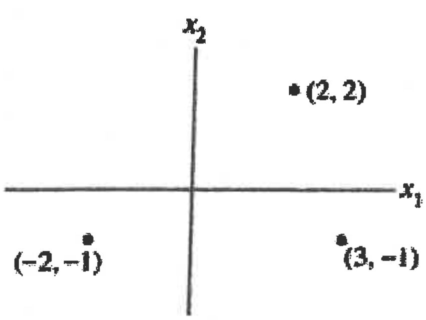
Ex 1: Given $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ find

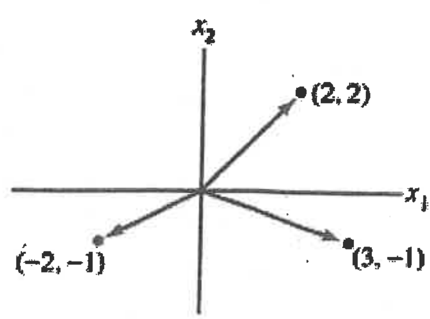
a) $\mathbf{u} + \mathbf{v}$

b) $3\mathbf{u}$

c) $2\mathbf{u} - 5\mathbf{v}$

Geometric Descriptions of $\mathbb{R}^2$

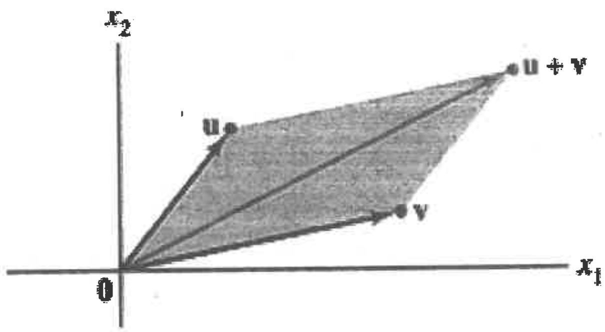




1.3: Vector Equations

Parallelogram Rule for Addition

If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See Figure 3.

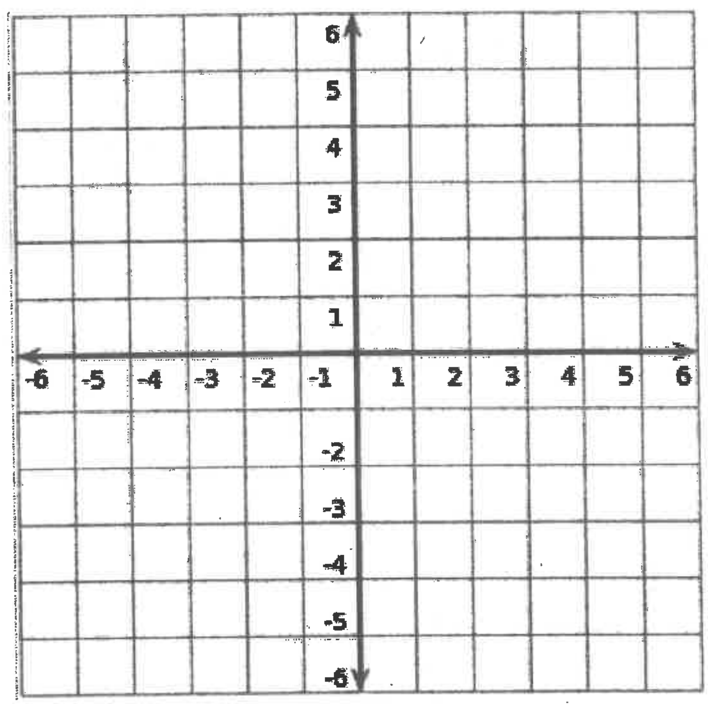


Ex 2: Given $\mathbf{u}=\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\2\end{bmatrix}$, draw their vectors and the following.

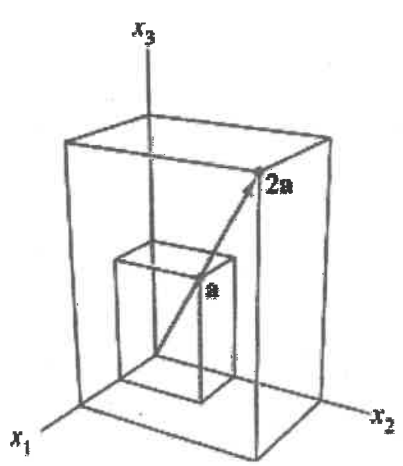
a) $\mathbf{u}+\mathbf{v}$

b) $3\mathbf{u}$

c) $-\frac{1}{2}\mathbf{v}$



Vectors in $\mathbb{R}^3$



Vectors in $\mathbb{R}^n$

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}$$

The Zero vector has entries of all zero, denoted by $\mathbf{0}$ or

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

Prove (i)

Claim: $u+v=v+u$

Proof.

Let $u,v\in\mathbb{R}^n$ be given.

Therefore $u+v=v+u$.

Prove (v)

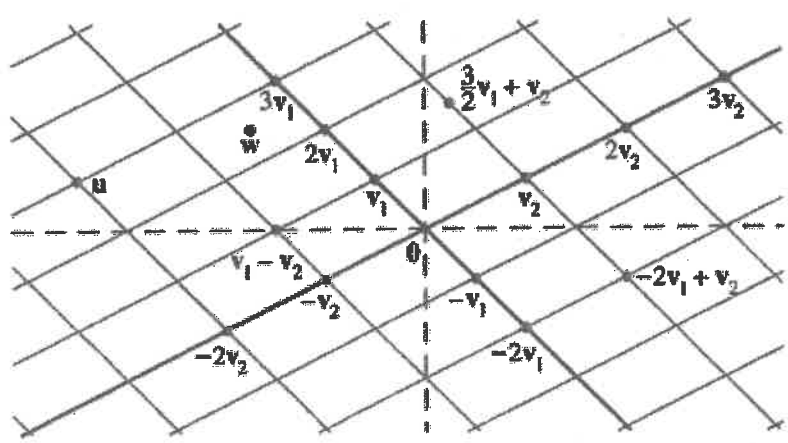
Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

Ex 3: The figure identifies selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



Ex 4: Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

1.3: Vector Equations

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}] \qquad (5)$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

Definition

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \ldots, \mathbf{v}_p$**. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

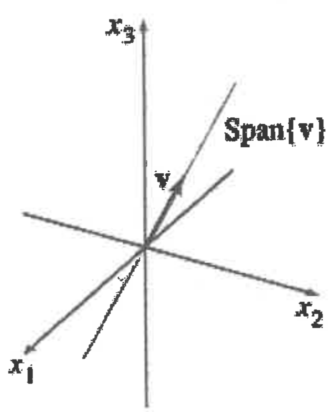
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

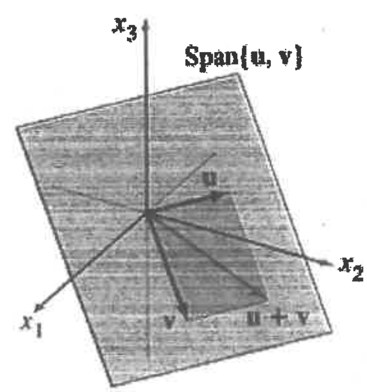
with $c_1, \ldots, c_p$ scalars.

$\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ means:

Every scalar multiple of individual vectors, $c\mathbf{v}_k$

Geometric Description of Span $\{\mathbf{v}\}$ and Span $\{\mathbf{u}, \mathbf{v}\}$





Ex 5: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 10 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$. $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane in $\mathbb{R}^3$.

Is **b** in that plane?

Ex 6: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$

For what value(s) of h is **y** in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$?

1.4: Matrix Equations

Math 220: Linear Algebra

Definition

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, **is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

$A\mathbf{x}$ is only defined if the number of ______________ of A equals the number of ______________ in $\mathbf{x}$.

Ex 1: Calculate the product $A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}_{2x3} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}_{3x1}$$

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}_{3x2} \begin{bmatrix} \quad \\ \quad \end{bmatrix}_{2x1}$$

1.4: Matrix Equations

Ex 2: For $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^3$ Write the linear combination of $5\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$ as a matrix times a vector.

Ex 3: Write the system of equations

$$\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \quad\quad - 5x_3 &= -2 \end{aligned}$$

as a

a) Vector Equation

b) Matrix Equation

c) Augmented matrix

1.4: Matrix Equations

Theorem 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \qquad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \qquad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \qquad (6)$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solutions if and only if **b** is a ________________

________________ of the columns of *A*.

Ex 4: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

1.4: Matrix Equations

Theorem 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

(Caveat/note: In Theorem 4, *A* is a coefficient matrix, not an augmented matrix.)

Ex 5: Compute $A\mathbf{x}=\mathbf{b}$ for $A=\begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & -3 \\ -3 & -2 & 5 \end{bmatrix}$ and $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Row–Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Ex 6: Compute

a) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} =$

b) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=$

(This is called the _______________ matrix, denoted by I)

If I_n represents nxn identity matrix, then $I_n\mathbf{x}=\mathbf{x}$ for every $\mathbf{x}\in\mathbb{R}^n$

Theorem 5

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

1.5: Solution Sets of Linear Systems

Math 220: Linear Algebra

A system of linear equations is called ______________________________ if it can be written as $A\mathbf{x}=0$ Such a system always has the ____________________ solution ______.

The important question is whether or not there is a ____________________________ solution to a homogeneous system.

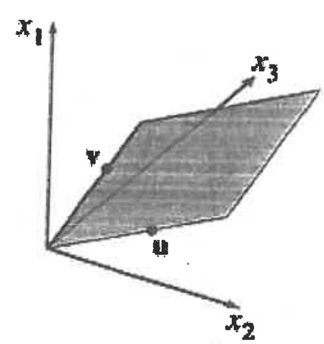
Since there is always a trivial solution, there is a non-trivial solution if and only if there is at least one __________ ____________________.

Ex 1: Determine whether the following has a non-trivial solution, and if so, describe the solution set.

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

Ex 2: Describe all the solutions of the homogeneous "system".

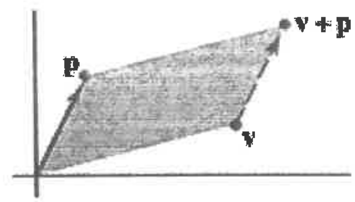
$$3x_1 - 4x_2 + 5x_3 = 0$$



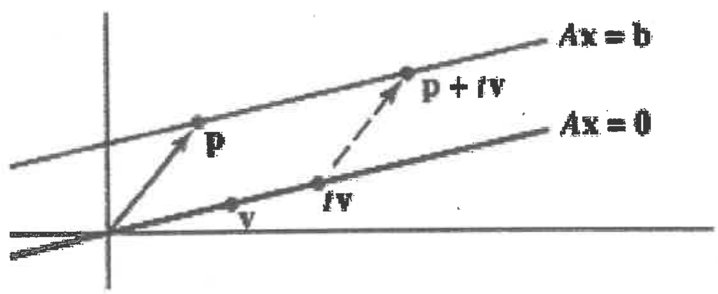
The previous example demonstrates how we can write solutions in Parametric Vector Form. $\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \in \mathbb{R})$

Solutions of Nonhomogeneous Systems

Ex 3: Describe all solutions of $A\mathbf{x} = \mathbf{b}$. $A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$



To visualize the solution set of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a ________________________.

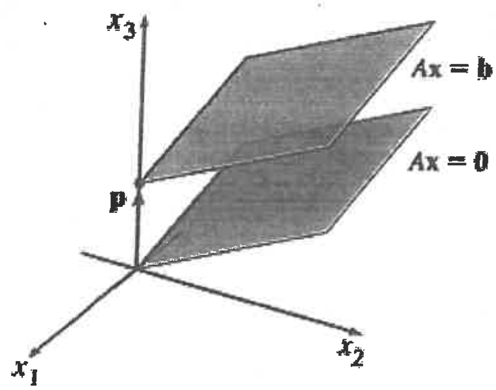


The solution set of $A\mathbf{x} = \mathbf{b}$ is a line through $\mathbf{p}$ ________________________ to the solution set of ____________.

1.5: Solution Sets of Linear Systems

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Claim (the first part of Theorem 6): Suppose that $\mathbf{p}$ is a solution of $\mathbf{Ax=b}$, so that $\mathbf{Ap=b}$. If $\mathbf{v}_h$ is any solution to the homogeneous equation $\mathbf{Ax=0}$ and $\mathbf{w=P+v}_h$ then $\mathbf{w}$ is a solution to $\mathbf{Ax=b}$.

Process: Writing a solution set (of a consistent system) in Parametric Vector Form.

1. **Row reduce the augmented matrix to reduced echelon form.**
2. **Express each basic variable in terms of any free variables appearing in an equation.**
3. **Write a typical solution x as a vector whose entries depend on the free variables, if any.**
4. **Decompose x into a linear combination of vectors (with numeric entries) using the free variables as parameters.**

1.5: Solution Sets of Linear Systems

Ex 4: Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9 \end{aligned}$$

Ex 5: Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form,

1.6 – Applications (read/review Network Flow as well – pages 53 – 54)

1.5: Solution Sets of Linear Systems

Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O \quad (4)$$

$$C_3H_8 : \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, O_2 : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, CO_2 : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, H_2O : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{Carbon} \\ \leftarrow \text{Hydrogen} \\ \leftarrow \text{Oxygen} \end{matrix}$$

1.7: Linear Independence

Math 220: Linear Algebra

Definition

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Ex 1: Determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. If not, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

1.7: Linear Independence

The columns of a matrix *A* are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Ex 2: Determine whether the columns of the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -3 & 1 & -2 \end{bmatrix}$ are linearly independent.

Ex 3: Are these sets linearly dependent (LD) or linearly independent (LI) and why?

The set	LD or LI	Why?
$\{\mathbf{v}\}$, not the zero vector		
$\{\mathbf{0}\}$		
$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$		
$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$		

1.7: Linear Independence

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Theorem 7 Characterization of Linearly Dependent Sets
An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Proof:

1.7: Linear Independence

Ex 4: Given the set of vectors $\{\mathbf{u},\mathbf{v},\mathbf{w}\}\in\mathbb{R}^3$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent, explain why vector $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ if and only if $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is linearly dependent.

Theorem 8
If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof:

Ex 5: Using Theorem 8, create a set of vectors in $\mathbb{R}^3$ that is linearly dependent, and don't automatically make some of the vectors obvious multiples or combinations of the others.

Theorem 9

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof:

Ex 6: Determine by inspection if the give set is linearly dependent.

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

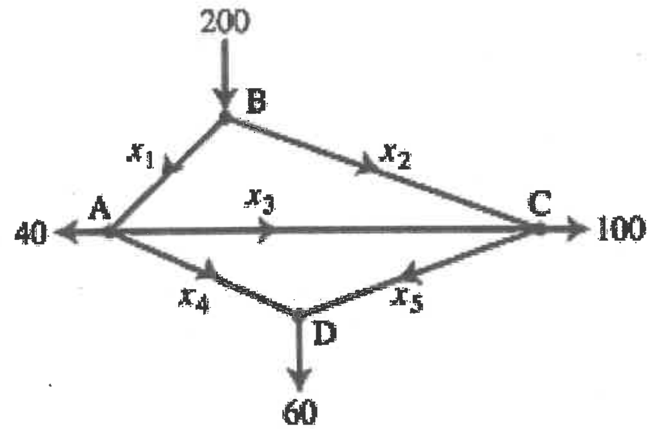
1.7: Linear Independence

Ex 7: Network flow exercise from 1.6 (we did a chemistry example previously).

a) Find the general traffic pattern in the freeway network shown in the figure. (Flow rates are in cars/minute)

b) Describe the general traffic pattern when the road whose flow is x_4 is closed.

c) When $x_4 = 0$, what is the minimum value of x_1?



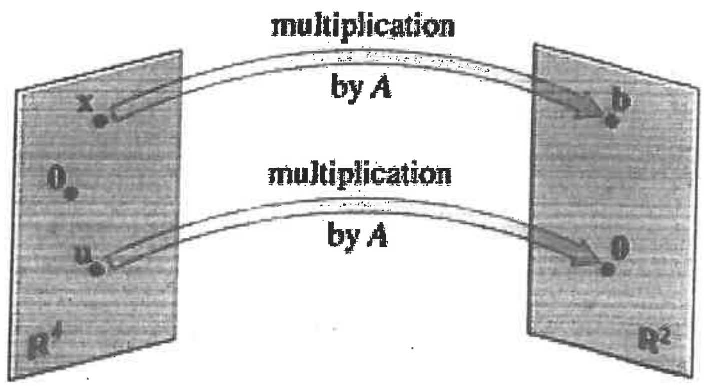
1.8: Linear Transformations

Math 220: Linear Algebra

While the matrix equation ____________ and the vector equation ______________________________ are essentially the same except for notation, there is a case where the matrix equation represents an action on a vector that isn't directly connected with a linear combination of vectors.

$$\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

$$\uparrow A \qquad \uparrow \mathbf{x} \qquad \uparrow \mathbf{b} \qquad\qquad \uparrow A \qquad \uparrow \mathbf{u} \qquad \uparrow$$



Does this picture look familiar from other math you've seen?

1.8: Linear Transformations

A ______________________ T from $\mathbb{R}^N$ to $\mathbb{R}^M$ is a rule that assigns each vector $\mathbf{x} \in \mathbb{R}^N$ to a vector $T(\mathbf{x}) \in \mathbb{R}^M$.

The set $\mathbb{R}^N$ is called the ______________ of T.

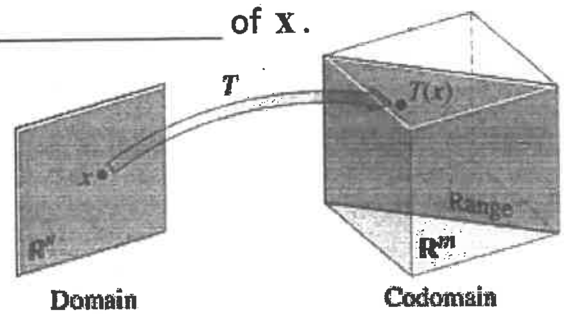
$$T : \mathbb{R}^N \to \mathbb{R}^M$$

The set $\mathbb{R}^M$ is called the ______________ of T.

For $\mathbf{x} \in \mathbb{R}^N$, the vector $T(\mathbf{x}) \in \mathbb{R}^M$ is called the ____________ of $\mathbf{x}$.

The set of all ______________ $T(\mathbf{x})$

is called the ______________ of T.



Review Ex. 5 on page 68 of a Rotation Transformation.

1.8: Linear Transformations

Ex 1: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$,

define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

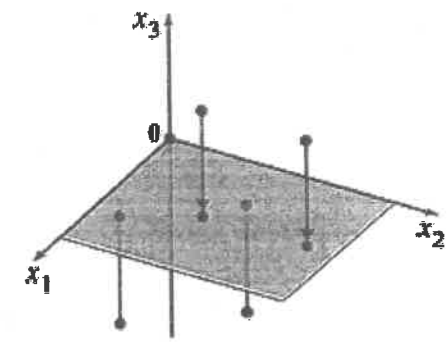
a. Find $T(\mathbf{u})$, the image of **u** under the transformation *T*.

b. Find an **x** in $\mathbb{R}^2$ whose image under *T* is **b**.

c. Is there more than one **x** whose image under *T* is **b**?

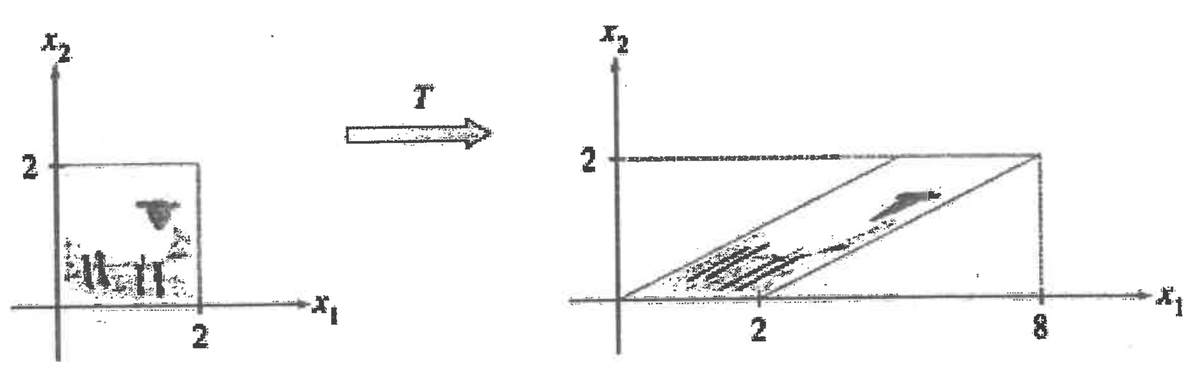
d. Determine if **c** is in the range of the transformation *T*.

Ex 2: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ *projects* points in $\mathbb{R}^3$ onto the x_1, x_2-plane because



Ex 3: Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a ____________ ________________________.

For the image below, let's look at the transformations of the vectors $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$



1.8: Linear Transformations

Definition

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all **u** in the domain of T.

Since the above properties are true for all matrices, then every ______________________ transformation is a ______________________ transformation. (Though the reverse is not true.)

Furthermore, (mini proof)

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors **u**, **v** in the domain of T and all scalars c, d.

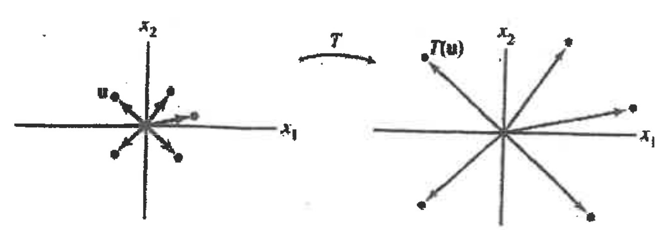
The second property here actually can be generalized to

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$$

This is referred to as a ______________________ ______________________ in engineering and physics.

Ex 4: Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. T is called a ______________________ when $0 \le r \le 1$ and a ______________________ when $r > 1$. Let $r = \pi$ and show that T is a linear transformation.

$T(c\mathbf{u} + d\mathbf{v}) =$



1.9: Matrix of a Linear Transformation

Math 220: Linear Algebra

Ex 1: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose *T* is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix}$.

Find a formula for the image of an arbitrary $\mathbf{x} \in \mathbb{R}^2$.

This shows us that knowing $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ can give us $T(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^2$. That is, for all $\mathbf{x} \in \mathbb{R}^2$ we have:

$$T(\mathbf{x}) = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$$

Theorem 10
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix *A* such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, *A* is the $m \times n$ matrix whose *j* th column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the *j* th column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \qquad (3)$$

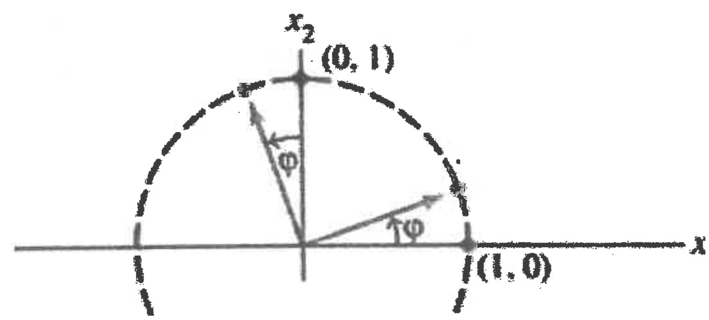
This Matrix *A* is called the ____

____.

1.9: Matrix of a Linear Transformation

Ex 2: Find the standard matrix *A* for the contraction transformation $T(\mathbf{x}) = \frac{1}{2}\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^2$.

Ex 3: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through the angle φ, with counterclockwise rotation for a positive angle (see the figure). Find the standard matrix *A* of this transformation.



1.9: Matrix of a Linear Transformation

Geometric Applications of Linear Transformations

Ex 4: Observe and discuss in the interactive ebook: *(also, pages 74-76)*

- Reflection

- Contraction & Expansion

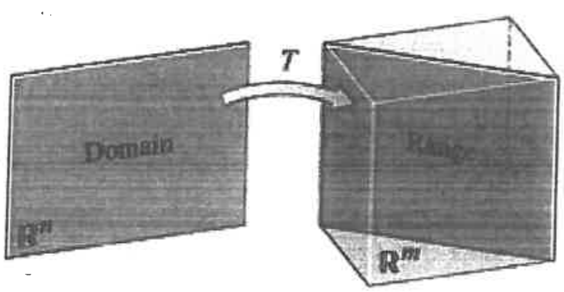
- Shear

- Projection

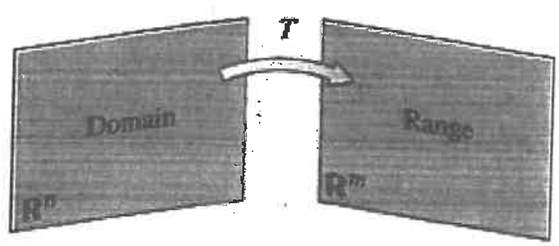
The Theory of Linear Transformations

Definition
A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is the image of *at least one* x in $\mathbb{R}^n$.

Another way of saying this is that the __________ of T is all of the ______________ $\mathbb{R}^m$



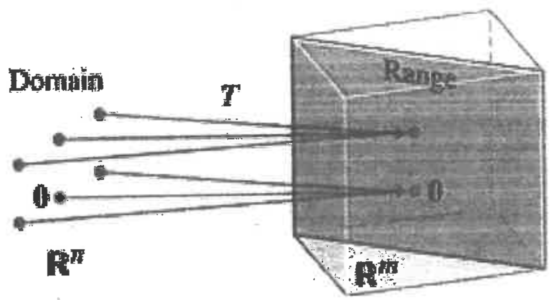
T is *not* onto $\mathbb{R}^m$



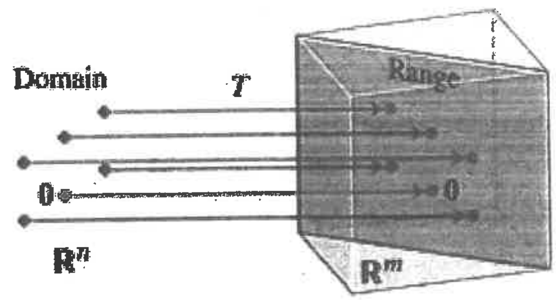
T is onto $\mathbb{R}^m$

1.9: Matrix of a Linear Transformation

Definition
A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.



T is *not* one-to-one



T is one-to-one

Theorem 11
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then *T* is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Proof.

Theorem 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let *A* be the standard matrix for *T*. Then:

a. *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of *A* span $\mathbb{R}^m$;

b. *T* is one-to-one if and only if the columns of *A* are linearly independent.

Proof.

Ex 5: Let T be the linear transformation whose standard matrix is below (2 cases). Determine whether they are "onto $\mathbb{R}^3$" and/or a one-to-one mapping.

a) $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

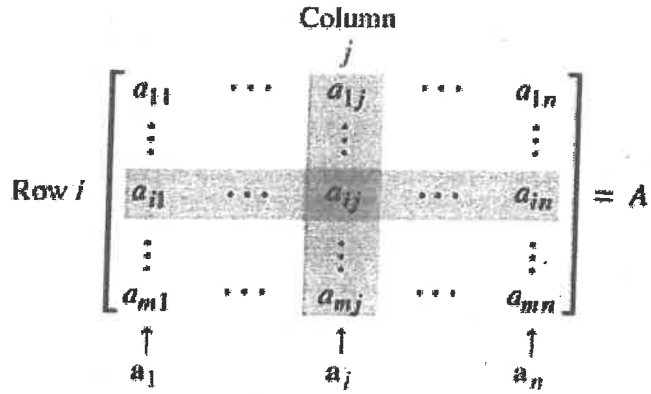
b) $B = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & 5 \end{bmatrix}$

	Why?	Why?
onto $\mathbb{R}^3$?		
one-to-one?		

2.1: Matrix Operations

Math 220: Linear Algebra

If *A* is an $m \times n$ matrix with m rows and n columns, then the entry in the ith row and jth column is denoted by ________ and is called the ____________________.



The ____________________ entries are $a_{11}, a_{22}, a_{33}, \ldots$ and they form the ____________________.

A ____________________ matrix is a square matrix ($n \times n$) whose non-diagonal entries are all ________________. The ____________________ matrix I_n is a diagonal matrix with ________ down the diagonal.

The ____________ matrix has all zeros in all of its entries and is written just as 0.

Two matrices are ________________ if they are the same ________ and the corresponding ____________ are ____________.

The __________ of two matrices ______________ is the ________ of their corresponding ____________. Thus, two matrices can only be ____________ if their __________ (   ) is the same. Otherwise, the sum is not defined.

Ex 1: Given $A=\begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

Find the following, if defined.

a) *A+B*

b) *B+C*

2.1: Matrix Operations

The _______________ _______________ _______ is the matrix whose entries are _____ times each entry of *A*.

The matrix _________ represents _________ and _________ is the same as _________.

Ex 2: Given $A=\begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find

a) 2*A*

b) *B*-2*A*

Theorem 1
Let *A*, *B*, and *C* be matrices of the same size, and let *r* and *s* be scalars.

a. $A+B=B+A$

b. $(A+B)+C=A+(B+C)$

c. $A+0=A$

d. $r(A+B)=rA+rB$

e. $(r+s)A=rA+sA$

f. $r(sA)=(rs)A$

Matrix Multiplication

Definition
If *A* is an $m \times n$ matrix, and if *B* is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product *AB* is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

Ex 3: Given $A=\begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, compute *CA*.

$C\mathbf{a}_1 =$

$C\mathbf{a}_2 =$

$C\mathbf{a}_3 =$

2.1: Matrix Operations

Ex 4: Given $A=\begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & -2 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, is the matrix AC defined?

Row–Column Rule for Computing *AB*
If the product *AB* is defined, then the entry in row *i* and column *j* of *AB* is the sum of the products of corresponding entries from row *i* of *A* and column *j* of *B*. If $(AB)_{ij}$ denotes the (i,j) -entry in *AB*, and if *A* is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Ex 5: Find the entries of the 3rd row of *AB*, where

$$A=\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \quad B=\begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

We could have just ignored the rest of *A* and computed

$$\boxed{\text{row}_i(AB) = \text{row}_i(A) \cdot B} \qquad \begin{bmatrix} 6 & -8 & -7 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

Theorem 2
Let *A* be an $m \times n$ matrix, and let *B* and *C* have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left distributive law)

c. $(B+C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

2.1: Matrix Operations

While the following properties are all true, be careful, the ____________________ property is not true, that is, *AB* _____ *BA*.

Ex 6: Let $A=\begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$. Show that these two matrices do not commute. That is, verify that $AB \neq BA$.

Warnings:

1. In general, $AB \neq BA$.

2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)

3. If a product *AB* is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

10. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$.

Verify that $AB = AC$ and yet $B \neq C$.

2.1: Matrix Operations

12. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

If A is an $n \times n$ matrix and if k is a positive integer, then $A^k =$

Given an $m \times n$ matrix A, then the ______________________ of A is the $n \times m$ matrix, denoted by ________ whose ______________________ are formed by the corresponding __________ of A.

Ex 7: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 2 & 4 \\ 6 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -4 & -5 \end{bmatrix}$. Find

$A^T =$ $B^T =$ $C^T =$

Theorem 3
Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

2.1: Matrix Operations

Practice Problems

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T, \mathbf{x}^T A^T, \mathbf{x}\mathbf{x}^T, \text{and } \mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

2. Let A be a 4×4 matrix and let **x** be a vector in $\mathbb{R}^4$. What is the fastest way to compute $A^2\mathbf{x}$? Count the multiplications.

3. Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB?

2.2: The Inverse of a Matrix

Math 220: Linear Algebra

Remember that the ______________________ ____________ or ______________ of a number, say 7 is ________ or ________. The actual definition of this is that

$$\cdot 7 = \qquad\qquad 7\cdot \quad =$$

An $(n \times n)$ matrix A is called __________________ if there is a matrix *C* such that

$$CA = I \qquad \text{and} \qquad AC = I$$

($I = I_n$ is the $n \times n$ identity matrix.)

Here, C is called the ____________ of A. Is C unique?

Yes, so denote the inverse with A^{-1} and

$$A^{-1}A = I \ \ \textbf{and} \ \ AA^{-1} = I$$

A matrix that is **NOT** invertible is called a __________________ matrix while a matrix that **IS** invertible is called a ____________________ matrix.

Ex 1: If $A = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix}$, verify that $C = A^{-1}$.

2.2: The Inverse of a Matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

This value $ad - bc$ is called the ______________________________ and we write

$$\det A = ad - bc$$

So theorem 4 states that ____ iff ____________.

Ex 2: Find the inverse of $A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$.

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof:

Ex 3: Use the inverse of the matrix $A=\begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$ from Ex 1 $\left(A^{-1}=\begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} \right)$

to solve the system $\begin{matrix} -2x_1-3x_2=5 \\ 3x_1+5x_2=-7 \end{matrix}$

Theorem 6

a. If *A* is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1}=A$$

b. If *A* and *B* are $n \times n$ invertible matrices, then so is *AB*, and the inverse of *AB* is the product of the inverses of *A* and *B* in the reverse order. That is,

$$(AB)^{-1}=B^{-1}A^{-1}$$

c. If *A* is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1}=(A^{-1})^T$$

Proofs:

2.2: The Inverse of a Matrix

From Theorem 6b, we can extrapolate to the following.

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

(Read pages 108-109 on Elementary Matrices)

We are going to look at finding the inverse of a matrix with a slightly different approach than this text.

If an $n \times n$ matrix *A* has an inverse, let's call that matrix *B*. Then

$$AB = I$$

This can be written as:

We can think of this as many systems, where each solution forms the columns vectors of our matrix B.

We could solve each one of these individually, or stack them all together.

Ex 4: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix}$.

Theorem 7
An $n \times n$ matrix *A* is invertible if and only if *A* is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces *A* to I_n also transforms I_n into A^{-1}.

Algorithm for Finding A^{-1}
Row reduce the augmented matrix $[A \quad I]$. If *A* is row equivalent to *I*, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, *A* does not have an inverse.

Ex 5: Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

(Do this by hand – more practice.)

2.3: Characteristics of Invertible Matrices

Math 220: Linear Algebra

Theorem 8 The Invertible Matrix Theorem

Let *A* be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given *A*, the statements are either all true or all false.

a. *A* is an invertible matrix.

b. *A* is row equivalent to the $n \times n$ identity matrix.

c. *A* has *n* pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of *A* form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each **b** in $\mathbb{R}^n$.

h. The columns of *A* span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix *C* such that $CA = I$.

k. There is an $n \times n$ matrix *D* such that $AD = I$.

l. A^T is an invertible matrix.

Theorem 5 from 2.2 could also make g. state ________________ solution.

If *A* and *B* are square matrices, and $AB=I$, then by j. and k. both *A* and *B* are invertible with $B=A^{-1}$ and $A=B^{-1}$.

2.3: Characteristics of Invertible Matrices

The Invertible Matrix Theorem essentially divides the set of all $n \times n$ matrices into two disjoint classes:

Invertible	Not Invertible

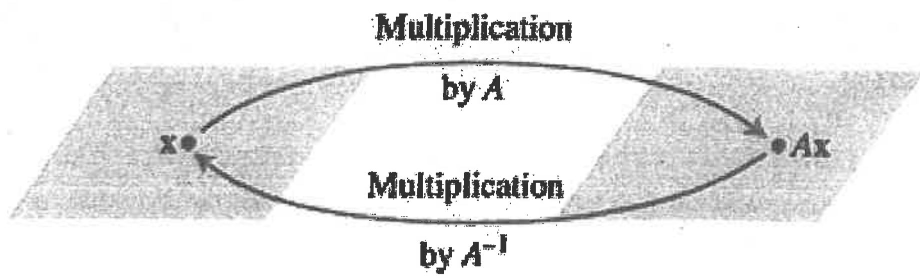
Ex 1: Use the Invertible Matrix Theorem to determine if the following are invertible.

$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ $B = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

Be careful, the Invertible Matrix Theorem only applies to ____________ matrices.

2.3: Characteristics of Invertible Matrices

If A is invertible, we can also think about ____________________ in light of linear transformations.



In general, a Linear Transformation $T:\mathbb{R}^N \to \mathbb{R}^N$ is ____________________ if there exists a function $S:\mathbb{R}^N \to \mathbb{R}^N$ such that

$$S(T(x)) = x \quad \text{for all } x \in \mathbb{R}^N$$
$$T(S(x)) = x \quad \text{for all } x \in \mathbb{R}^N$$

We call S the ____________________ of T and write it as ____________.

Theorem 9

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

Ex 2: What can be said about a one-to-one linear transformation $T:\mathbb{R}^N \to \mathbb{R}^N$?

Practice Problems

2. Suppose that for a certain $n \times n$ matrix *A*, statement (g) of the Invertible Matrix Theorem is *not* true. What can you say about equations of the form $A\mathbf{x} = \mathbf{b}$?

3. Suppose that *A* and *B* are $n \times n$ matrices and the equation $AB\mathbf{x} = \mathbf{0}$ has a nontrivial solution. What can you say about the matrix *AB*?

3.1 & 3.2: Determinants

Math 220: Linear Algebra

Although out of fashion, determinants played a large role in the early development of linear algebra. Four uses of determinants include the following: Determinants help us "determine" if a system of linear equations has a unique solution. They are a mechanism to "determine" whether the inverse of a matrix exists (this would have come later). They may be geometrically interpreted as the scaling factor of a linear transformation. And the determinant is also a calculating mechanism used elsewhere in math to find things such as the cross-product (Calculus III), Jacobian (Calculus IV), and the Wronskian (Differential Equations).

As to why they have fallen out of favor? Well they are computationally expensive even with modern technology. So we have adopted other ways to accomplish their original purpose.

Their primary reason for being in this course is that they are needed for our development of the eigenvalue and eigenvector in a subsequent chapter.

Ex 1: If $A = \begin{bmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ find $\det A$ which is also notated $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix}$

Ex 2: Calculate $\begin{vmatrix} 0 & 1 & 4 \\ 5 & 0 & -3 \\ 2 & 1 & 3 \end{vmatrix}$ by expanding across the second column.

3.1 & 3.2: Determinants

Ex 3: Compute the determinant: $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -4 \end{vmatrix}$

Theorem 2
If *A* is a triangular matrix, then det *A* is the product of the entries on the main diagonal of *A*.

Ex 4: Compute the determinant: $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$

Theorem 3: Row Operations

Let *A* be a square matrix

a. If a multiple of one row of *A* (old) is added to another row to produce a matrix *B* (new), then $\det A = \det B$.
b. If two rows of *A* (old) are interchanged to produce *B* (new), then $\det A = -\det B$.

c. If one row of A (old) is multiplied by k to produce B (new), then $\det A = \frac{1}{k}\det B$

Ex 5: Find the determinant by first row-reducing to echelon form.

$$\begin{vmatrix} 3 & 3 & -3 \\ 2 & -3 & -5 \\ 3 & 4 & -4 \end{vmatrix}$$

Ex 6: Find the determinant by first row-reducing to echelon form.

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$$

Let's consider two different triangular matrices and their invertibility. The focus on triangular matrices is reasonable as we learned in a previous section that row operations do not impact the invertibility of matrices.

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \qquad U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\det U \neq 0 \qquad\qquad \det U = 0$$

Theorem 4

A square matrix A is invertible if and only if $\det A \neq 0$.

Ex 7: Revisiting (Ex 6:), at what point could we have stopped?

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$$

Theorem 5

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Theorem 6 Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Ex 8: Verify Thm 6 for $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix}$

3.1 & 3.2: Determinants

Practice Problems

1. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.

2. Use a determinant to decide if $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

3. Let A be an $n \times n$ matrix such that $A^2 = I$. Show that $\det A = \pm 1$.

4.1: Vector Spaces and Subspaces

Math 220: Linear Algebra

Definition

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in V and for all scalars c and d.

1. The sum of **u** and **v**, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a zero vector **0** in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each **u** in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of **u** by c, denoted by c **u**, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

It also follows that

$$\begin{aligned} 0\mathbf{u} &= \mathbf{0} \quad (1)\\ c\mathbf{0} &= \mathbf{0} \quad (2)\\ -\mathbf{u} &= (-1)\mathbf{u} \quad (3) \end{aligned}$$

The spaces ________________ for $n \geq 1$ are the best examples of vector spaces. We will picture ______________ for much of our discussion of vector spaces.

4.1: Vector Spaces and Subspaces

Ex 1:

Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule (from Section 1.3), and for each **v** in V, define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of **v**, pointing in the same direction as **v** if $c \geq 0$ and otherwise pointing in the opposite direction. (See Figure 1.) Show that V is a vector space. This space is a common model in physical problems for various forces.

Read Example 3 on page 193

Ex 2: Discuss whether the set P_n of polynomials of degree at most n is a vector space.

Read Example 5 on page 194

Definition

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. H is closed under vector addition. That is, for each **u** and **v** in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each **u** in H and each scalar c, the vector c **u** is in H.

Note: Every subspace is itself a Vector space.

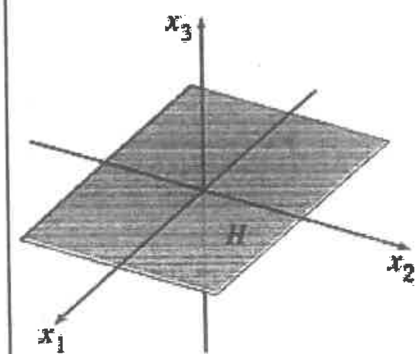
The set of just the ________ vector in a vector space V is a subspace of V called the ____________ ______________________ and written _________.

Ex 3: Discuss that P, set of all polynomials and a subspace of the set of all real-valued functions, and P_n is a subspace of *P*.

What about a plane not through the origin? Or a line in $\mathbb{R}^2$ not through the origin? Are they Subspaces? (of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively).

Ex 4: The vector space $\mathbb{R}^2$ is NOT a subspace of $\mathbb{R}^3$, but *H* is. Discuss.

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$



Ex 5: Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space *V*, let $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that *H* is a subspace of *V*.

Theorem 1
If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space *V*, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of *V*.

- We call this subspace the ________________ ______________________ by $\{v_1, \ldots v_p\}$

And for any subspace *H*, we call the set $\{v_1, \ldots v_p\}$ such that $H = \text{Span}\{v_1, \ldots v_p\}$,

the __________________ _________.

Ex 6: Let H be the set of all vectors of the form $\begin{bmatrix} a \\ 3a+b \\ b \\ a-2b \end{bmatrix}$ where *a* and *b* are arbitrary scalars. Show that *H* is a subspace of $\mathbb{R}^4$

We can think of the vectors in a spanning set as the "handles" that define a subspace H, and allow us to hold it and work with it.

Ex 7: For what value(s) of *h* will **y** be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

(This is the same example in the text from 1.3 – now with the context of subspaces.)

Practice Problems

1. Show that the set *H* of all points in $\mathbb{R}^2$ of the form $(3s, 2 + 5s)$ is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector **u** in *H* and a scalar *c* such that *c* **u** is not in *H*.)

3. An $n \times n$ matrix *A* is said to be symmetric if $A^T = A$. Let *S* be the set of all 3×3 symmetric matrices. Show that *S* is a subspace of $M_{3\times3}$, the vector space of 3×3 matrices.

4.2: Null & Col Spaces and Linear Transformations

Math 220: Linear Algebra

Remember that a homogeneous system of equations

$$\begin{aligned} 5x_1 + 21x_2 + 19x_3 &= 0 \\ 13x_1 + 23x_2 + 2x_3 &= 0 \\ 8x_1 + 14x_2 + x_3 &= 0 \end{aligned}$$

can be written in matrix form as $A\mathbf{x} = \mathbf{0}$ where

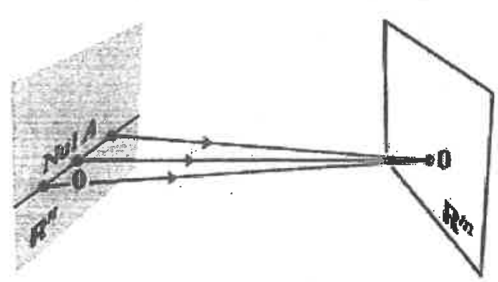
$$A = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

The solution set is all the vectors $\mathbf{x}$ that satisfy the matrix equation. We are going to name this set of solutions the ________ ____________.

Definition
The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Ex 1: Let A be the matrix defined above. Determine whether the vector $\mathbf{u} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ belongs to the null space of A.



4.2: Null & Col Spaces and Linear Transformations

Theorem 2
The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Proof:

Ex 2: Let H be the set of vectors in $\mathbb{R}^3$ whose coordinates a, b, and c satisfy the equations ______________________ and ______________________.
Show that H is a subspace of $\mathbb{R}^3$. *(Hint: Create two dependence relations.)*

Ex 3: Find a spanning set for the null space of the matrix $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$.

Two properties of null spaces that contain nonzero vectors that we see from the last example.

1. The spanning set generated using the previous method is automatically ________________ ________________.
2. The number linearly independent vectors in the spanning set of Nul A equals the number of __________ ________________ in the equation $A\mathbf{x}=\mathbf{0}$.

Definition
The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

Theorem 3
The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

Ex 4: Find a matrix A such that $W = \text{Col}\, A$.

$$W = \left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} : b, c, d \text{ real} \right\}$$

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

4.2: Null & Col Spaces and Linear Transformations

Ex 5: Given the matrix $A=\begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 4 \\ 1 & 2 & 4 & -1 \end{bmatrix}$, answer the following.

a) Find $\mathbb{R}^k$ such that Nul A is a subspace of $\mathbb{R}^k$.

b) Find $\mathbb{R}^k$ such that Col *A* is a subspace of $\mathbb{R}^k$.

c) Find an example of a nonzero vector in Nul *A* as well as Nul *A*.

d) Find a nonzero vector in Col *A*.

e) Is $\begin{bmatrix} 1 \\ 4 \\ -2 \\ 1 \end{bmatrix}$ in the Nul *A*? Is $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$ in the Nul *A*?

f) Is $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ in Col *A*?

4.2: Null & Col Spaces and Linear Transformations

Contrast Between Nul *A* and Col *A* for an $m \times n$ Matrix *A*

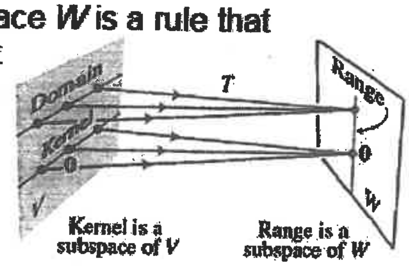
Nul *A*	**Col *A***
1. Nul *A* is a subspace of $\mathbb{R}^n$.	**1.** Col *A* is a subspace of $\mathbb{R}^m$.
2. Nul *A* is implicitly defined; that is, you are given only a condition $(A\mathbf{x} = \mathbf{0})$ that vectors in Nul *A* must satisfy.	**2.** Col *A* is explicitly defined; that is, you are told how to build vectors in Col *A*.
3. It takes time to find vectors in Nul *A*. Row operations on $[A \;\; \mathbf{0}]$ are required.	**3.** It is easy to find vectors in Col *A*. The columns of *A* are displayed; others are formed from them.
4. There is no obvious relation between Nul *A* and the entries in *A*.	**4.** There is an obvious relation between Col *A* and the entries in *A*, since each column of *A* is in Col *A*.
5. A typical vector **v** in Nul *A* has the property $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector **v** in Col *A* has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector **v**, it is easy to tell if **v** is in Nul *A*. Just compute *A* **v**.	**6.** Given a specific vector **v**, it may take time to tell if **v** is in Col *A*. Row operations on $[A \;\; \mathbf{v}]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every **b** in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Definition

A **linear transformation** *T* from a vector space *V* into a vector space *W* is a rule that assigns to each vector **x** in *V* a unique vector *T* **(x)** in *W*, such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in *V*, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all **u** in *V* and all scalers *c*.



The null space of a linear transformation is called the ____________ and is the set if all vectors $\mathbf{u} \in V$ such that $T(\mathbf{u}) = \mathbf{0}$.

The ____________ of T is the set of all vectors in *W* of the form $T(\mathbf{x})$ for some $\mathbf{x} \in V$.

Ex 6:

(Calculus required) Let *V* be the vector space of all real-valued functions *f* defined on an interval $[a,b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a,b]$. Let *W* be the vector space $C[a,b]$ of all continuous functions on $[a,b]$, and let $D: V \to W$ be the transformation that changes *f* in *V* into its derivative f'. In calculus, two simple differentiation rules are

$$D(f+g) = D(f) + D(g) \quad \text{and} \quad D(cf) = cD(f)$$

That is, *D* is a linear transformation. It can be shown that the kernel of *D* is the set of constant functions on $[a,b]$ and the range of *D* is the set *W* of all continuous functions on $[a,b]$.

Practice Problems

1. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show in two different ways that *W* is a subspace of $\mathbb{R}^3$. (Use two theorems.)

2. Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose you know that the equations $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = \mathbf{w}$ are both consistent. What can you say about the equation $A\mathbf{x} = \mathbf{v} + \mathbf{w}$?

4.3: Linearly Independent Sets; Bases

Math 220: Linear Algebra

Recall the previous definitions of Linearly Independent and Linearly Dependent. We are now going to think in terms of a Vector Space *V*, rather than just $\mathbb{R}^n$.

> **Definition**
> An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in ~~$\mathbb{R}^n$~~ V is said to be **linearly independent** if the vector equation
>
> $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$
>
> has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

And recall that

Theorem 4
An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

If a vector space is not $\mathbb{R}^n$ described with the easily solved matrix equation $A\mathbf{x} = \mathbf{0}$, then we need Theorem 4 to show a linear dependence relation to prove linear dependence.

Ex 1: Discuss the linear dependence or independence of the following sets on $C[0,1]$, the space of all continuous functions on $0 \le t \le 1$.

$\{\sin t, \cos t\}$

$\{\sin t \cos t, \sin 2t\}$

4.3: Linearly Independent Sets; Bases

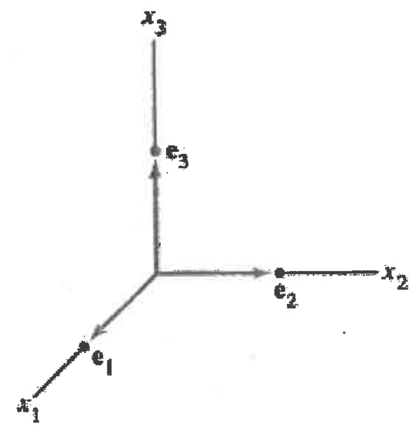
Definition
Let H be a subspace of a vector space V. An indexed set of vectors $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) B is a linearly independent set, and

(ii) the subspace spanned by B coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

Ex 2: What can we say about an invertible n x n matrix A?



The columns of the identity matrix, $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$ is called the ______________ ______________ for $\mathbb{R}^n$.

Ex 3: Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

Do $\{\mathbf{v}_1, \mathbf{v}_2\}$ **form a basis for** $\mathbb{R}^2$?

Ex 4: Let $S = \{1, t, t^2, \ldots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$.

A basis is an "efficient" spanning set because it contains no unnecessary vectors.

Ex 5: Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as in Ex 3. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

Theorem 5 The Spanning Set Theorem
Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$ —is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Proof:

4.3: Linearly Independent Sets; Bases

We already know how to find a basis for the Nul A, as we saw that the row reduced system that describes the solutions of Nul A, is already linearly independent.

However, finding a basis for Col A that doesn't have unneeded vectors is our next step.

Ex 6: Find a Basis for Col B where

$$B=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5\end{bmatrix}=\begin{bmatrix}1 & 0 & -3 & 0 & 4\\ 0 & 1 & -4 & 0 & -5\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Ex 7: Find a Basis for Col A where, A reduces to the matrix B in the previous example.

$$A=\begin{bmatrix}1 & 0 & -3 & 1 & 2\\ 0 & 1 & -4 & -3 & 1\\ -3 & 2 & 1 & -8 & -6\\ 2 & -3 & 6 & 7 & 9\end{bmatrix}$$

Since $A\mathbf{x}=\mathbf{0}$ and the reduced echelon form $B\mathbf{x}=\mathbf{0}$ have the exact same solution sets, then their columns have the exact same dependence relationships. Let's check.

4.3: Linearly Independent Sets; Bases

WARNING: You must use the original pivot columns of A.

Question: Why doesn't $\text{Col}\, A = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$?

Theorem 6
The pivot columns of a matrix A form a basis for Col A.

A basis is basically the smallest spanning set possible. Remove any vectors from it, and the set is no longer spanned, add any vectors to it, and it becomes linearly dependent.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$ — **A basis for $\mathbb{R}^3$** — **Spans $\mathbb{R}^3$ but is linearly dependent**

Practice Problems

1. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-2\\7\\-9\end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

4.3: Linearly Independent Sets; Bases

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace *W* spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in *H* is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for *H*?

4.4: Coordinate Systems

Math 220: Linear Algebra

Theorem 7 The Unique Representation Theorem
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space *V*. Then for each **x** in *V*, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

Proof:

Definition
Suppose $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for *V* and **x** is in *V*. The **coordinates of x relative to the basis** *B* (or the ***B*-coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

We call this vector the ______________ ________

____ ____ (______________ ____ ______ _____)

or the ______________________ __________ _____ _____

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\mathbf{x} \mapsto [\mathbf{x}]_B$ is the ______________ ______________(determined by *B*)

Ex 1: Consider a basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Suppose an **x** in $\mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find **x**.

4.4: Coordinate Systems

Ex 2: The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of **x** relative to the *standard basis* $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, since

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \qquad\qquad\qquad =$$

If $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, then $[\mathbf{x}]_\varepsilon = \mathbf{x}$.

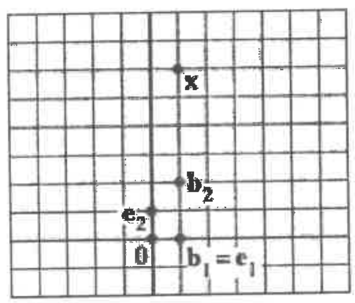


FIGURE 1 Standard graph paper.

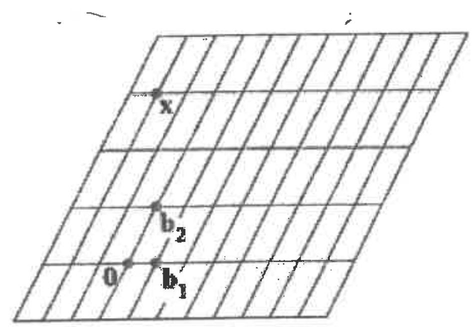
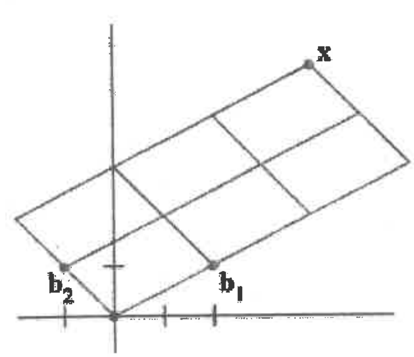


FIGURE 2 *B*-graph paper.

See Example 3 on page 219.

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_B$ of **x** relative to *B*.



The matrix P_B changes the B-coordinates of a vector **x** into the standard coordinates for **x**. An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Let

$$\boxed{P_B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \cdots \ \ \mathbf{b}_n]}$$

Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

is equivalent to

$$\boxed{\mathbf{x} = P_B[\mathbf{x}]_B} \qquad (4)$$

We call P_B the **change-of-coordinates matrix** from B to the standard basis in $\mathbb{R}^n$. Left-multiplication by P_B transforms the coordinate vector $[\mathbf{x}]_B$ into **x**.

Since the columns of P_B form a basis, they are linearly independent, and have an inverse, which leads to

$$P_B^{-1}\mathbf{x} = [\mathbf{x}]_B$$

The Coordinate Mapping

Choosing a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space V introduces a coordinate system in V. The coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ connects the possibly unfamiliar space V to the familiar space $\mathbb{R}^n$. See Figure 5. Points in V can now be identified by their new "names."

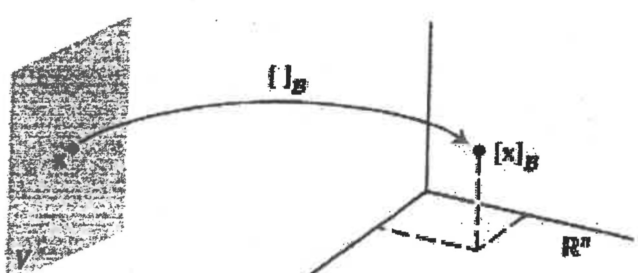


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

4.4: Coordinate Systems

Theorem 8
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

A one-to-one linear transformation from a vector space V onto a vector space W is called an ______________________ from V onto W.

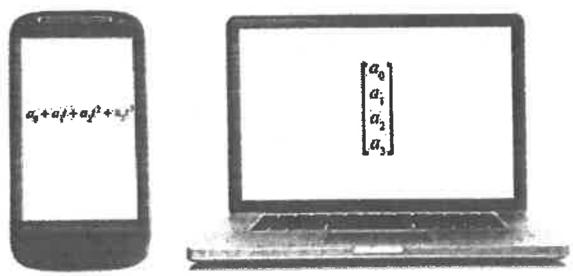
Essentially, these two vector spaces are indistinguishable.

Ex 4: Let B be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, let $B = \{1, t, t^2, t^3\}$. A typical element p of $\mathbb{P}_3$ has the form

$$\mathbf{p}(t) =$$

Since p is a linear combination of the standard basis vectors, then $[\mathbf{p}]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

So $\mathbf{p} \mapsto [\mathbf{p}]_B$ is an isomorphism from $\mathbb{P}_3$ onto $\mathbb{R}^4$.



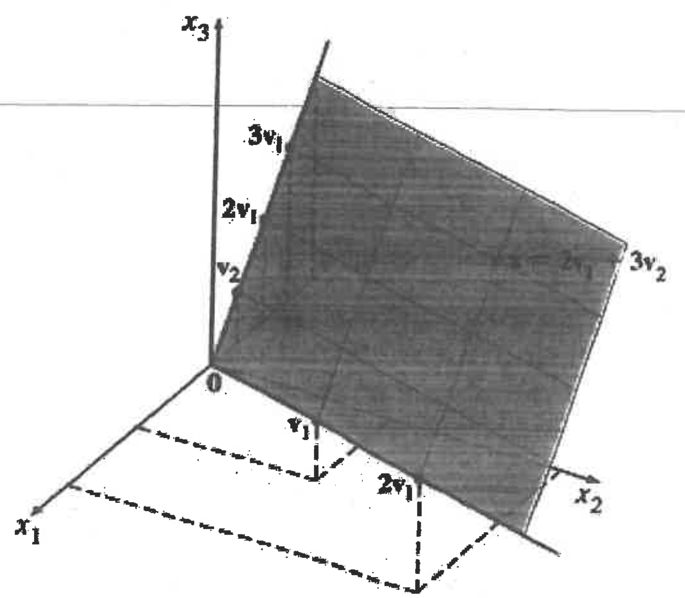
Ex 5: Use coordinate vectors to test the linear independence of the sets of polynomials.

a) $1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3$

b) Is this a basis $\mathbb{P}_3$? $(1-t)^2, t-2t^2+t^3, (1-t)^3$

Ex 6: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$,

and $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then B is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if **x** is in H, and if it is, find the coordinate vector of **x** relative to B.



4.4: Coordinate Systems

Practice Problems

1. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.

a. Show that the set $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$.

b. Find the change-of-coordinates matrix from B to the standard basis.

c. Write the equation that relates $\mathbf{x}$ in $\mathbb{R}^3$ to $[\mathbf{x}]_B$.

d. Find $[\mathbf{x}]_B$, for the $\mathbf{x}$ given above.

2. The set $B = \{1+t, 1+t^2, t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6 + 3t - t^2$ relative to B.

4.5: The Dimension of a Vector Space, Rank

Math 220: Linear Algebra

Intro: These sections focus on a number of characteristics of common subspaces: dimension, rank, nullity, and the row space.

Theorem 9

If a vector space *V* has a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in *V* containing more than *n* vectors must be linearly dependent.

Theorem 10

If a vector space *V* has a basis of *n* vectors, then every basis of *V* must consist of exactly *n* vectors.

Definition

If *V* is spanned by a finite set, then *V* is said to be **finite-dimensional,** and the **dimension** of *V*, written as dim *V*, is the number of vectors in a basis for *V*. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If *V* is not spanned by a finite set, then *V* is said to be **infinite-dimensional.**

Ex 1: Find the following

a) $\dim \mathbb{R}^n =$ _____

b) $\dim P_3 =$ _____ $\left(\text{recall } P_3 = \text{Span}\{1, t, t^2, t^3\}\right)$

c) $\dim P_n =$ _____

d) $\dim P$ ____________________________ (recall *P* = all polynomials)

e) Given $H = \text{span}\left\{\begin{bmatrix} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \end{bmatrix}\right\}$ we can see $\dim H =$

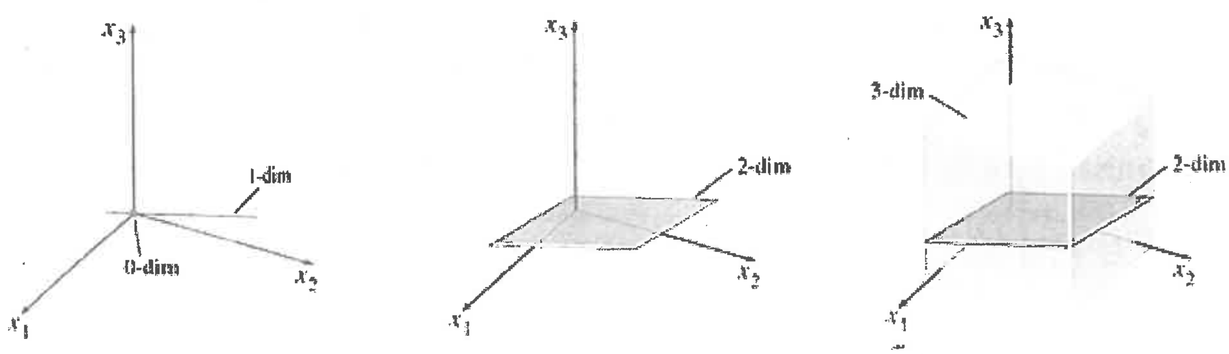
f) Given $G = \text{span}\left\{\begin{bmatrix} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \end{bmatrix}\right\}$ we can see $\dim G =$

4.5: The Dimension of a Vector Space, Rank

Ex 2: Find the dimension of the subspace

$$\left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : a,b,c \text{ in } \mathbb{R} \right\}$$

The subspaces of $\mathbb{R}^3$ can be classified by dimension now.



Theorem 11

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Proof:

4.5: The Dimension of a Vector Space, Rank

Theorem 12 The Basis Theorem
Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Proof:

What can we say about the dimension of Col A and Nul A?

The dimension of the null space of A is

The dimension of the column space of A is:

Ex 3: Determine the dimensions of the null space and the column space of A.

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.5: The Dimension of a Vector Space, Rank

Row Space

The set of all the linear combinations of the row vectors of an $m \times n$ matrix *A* is called the ________ ________________ of *A*, and is denoted by ______________. Since there are *n* entries in each row, Row *A* is a subspace of $\mathbb{R}^n$. Also, Row *A* = ______________.

Ex 4: Find a spanning set for Row *A*.

$$A=\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix}$$

Theorem 13
If two matrices *A* and *B* are row equivalent, then their row spaces are the same. If *B* is in echelon form, the nonzero rows of *B* form a basis for the row space of *A* as well as for that of *B*.

Ex 5: Find bases for the row space, column space, and null space of A.

$$A=\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.5: The Dimension of a Vector Space, Rank

The ____________ of A is the dimension of the column space of A.

The ____________ of ______ is the dimension of the row space of A.

The ____________ of A is the dimension of the null space of A (though this text just uses ________________.)

Theorem 14 The Rank Theorem
The dimensions of the column space and the row space of an $m \times n$ matrix *A* are equal. This common dimension, the rank of *A*, also equals the number of pivot positions in *A* and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

(See proof on page 235.)

Ex 6: a) If A is an _____x______ matrix with three-dimensional null space, what is the rank of A?

b) Could a 3x5 matrix have a one-dimensional null space?

In chapter 6 we will learn that Row A and Nul A have only the ____________ ________________ in common, and they are actually ______________________ to each other. ***Take a look at example 4 on page 236.***

Ex 7: A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

4.5: The Dimension of a Vector Space, Rank

Theorem The Invertible Matrix Theorem (continued)
Let *A* be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that *A* is an invertible matrix.

m. The columns of *A* form a basis of $\mathbb{R}^n$.

n. $\text{Col } A = \mathbb{R}^n$

o. $\dim \text{Col } A = n$

p. $\text{rank } A = n$

q. $\text{Nul } A = \{\mathbf{0}\}$

r. $\dim \text{Nul } A = 0$

Practice Problems
The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank *A* and dim Nul *A*.

2. Find bases for Col *A* and Row *A*.

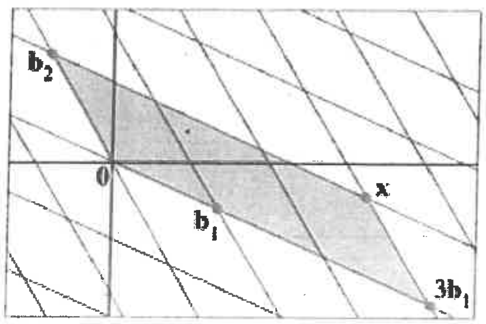
3. What is the next step to perform to find a basis for Nul *A*?

4. How many pivot columns are in a row echelon form of A^T?

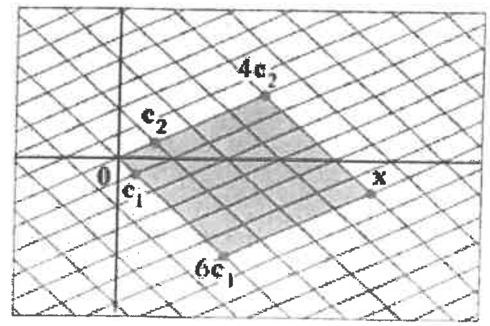
4.6: Change of Basis
Math 220: Linear Algebra

We are now going to look at converting a vector x in one coordinate system into another coordinate system – same vector, different coordinate representation.

Consider the following vector spaces spanned by $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{c}_1, \mathbf{c}_2\}$ respectively.



(a)



(b)

By observation, find $[\mathbf{x}]_B =$ and $[\mathbf{x}]_C =$

Ex 1: Consider two bases $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$ (that is, $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$), find $[\mathbf{x}]_C$.

4.6: Change of Basis

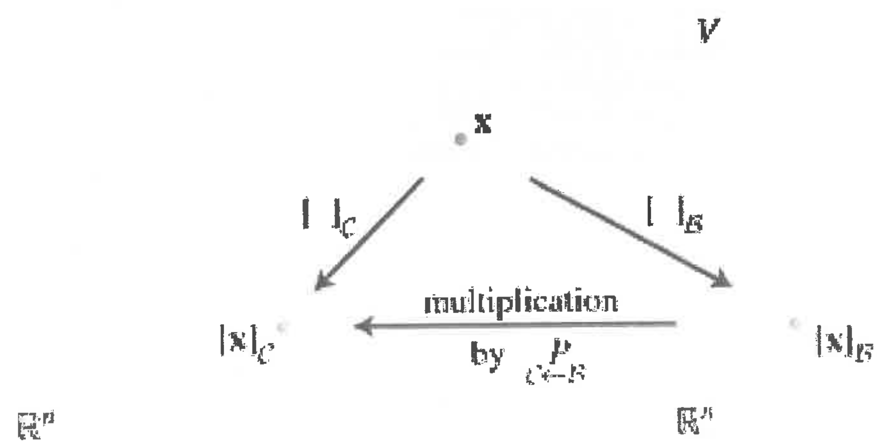
Theorem 15
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{C \leftarrow B}{P}$ such that

$$[\mathbf{x}]_C = \underset{C \leftarrow B}{P}[\mathbf{x}]_B \qquad (4)$$

The columns of $\underset{C \leftarrow B}{P}$ are the C-coordinate vectors of the vectors in the basis B. That is,

$$\underset{C \leftarrow B}{P} = [[\mathbf{b}_1]_C \quad [\mathbf{b}_2]_C \quad \cdots \quad [\mathbf{b}_n]_C] \qquad (5)$$

$\underset{C \leftarrow B}{P}$ is the ____



Why are the columns of $\underset{C \leftarrow B}{P}$ linearly independent?

So $\underset{C \leftarrow B}{P}$ is ____________________________.

So equation (4) above can be re-written as $\qquad [\mathbf{x}]_C = \quad [\mathbf{x}]_B$

Since $\underset{C \leftarrow B}{P}$ is the matrix that converts B-coordinates to C-coordinates, what should $\left(\underset{C \leftarrow B}{P}\right)^{-1}$ do?

$$\left(\underset{C\leftarrow B}{P}\right)^{-1} = \underset{B\leftarrow C}{P}$$

Change of Basis in $\mathbb{R}^n$

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and ε is the *standard basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_\varepsilon = \mathbf{b}_1$, and likewise for the other vectors in *B*. In this case, $\underset{\varepsilon\leftarrow B}{P}$ is the same as the change-of-coordinates matrix P_B introduced in Section 4.4, namely,

$$P_B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

However, to change coordinates between two non-standard bases in $\mathbb{R}^n$, we will need to use Theorem 15, and find coordinate vectors of the ________ ____________ relative to the ________ ____________.

Ex 2:

Let $\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from *B* to *C*.

4.6: Change of Basis

$$\left[\, \mathbf{c}_1 \;\; \mathbf{c}_2 \mid \mathbf{b}_1 \;\; \mathbf{b}_2 \,\right] \sim \left[\, I \mid \underset{C\leftarrow B}{P} \,\right]$$

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from C to B.

b. Find the change-of-coordinates matrix from B to C.

Practice Problems

1. Let $F = \{\mathbf{f}_1, \mathbf{f}_2\}$ and $G = \{\mathbf{g}_1, \mathbf{g}_2\}$ be bases for a vector space V, and let P be a matrix whose columns are $[\mathbf{f}_1]_G$ and $[\mathbf{f}_2]_G$. Which of the following equations is satisfied by P for all $\mathbf{v}$ in V?

(i) $[\mathbf{v}]_F = P[\mathbf{v}]_G$

(ii) $[\mathbf{v}]_G = P[\mathbf{v}]_F$

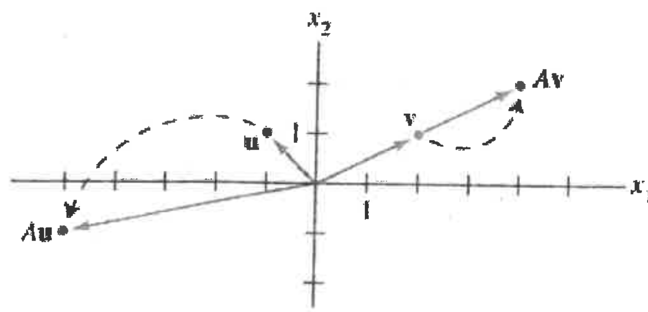
2. Let B and C be as in Example 1. Use the results of that example to find the change-of-coordinates matrix from C to B.

5.1: Eigenvectors and Eigenvalues

Math 220: Linear Algebra

Ex 1: Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Calculate $A\mathbf{u}$ and $A\mathbf{v}$.

What do you notice about either of them?



Definition

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathbf{x} = \lambda\mathbf{x}$; such an **x** is called an *eigenvector corresponding to* λ.

Ex 2: Is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? If so, find the eigenvalue.

Is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? If so, find the eigenvalue.

5.1: Eigenvectors and Eigenvalues

Ex 3: Show that 5 is an eigenvalue of the matrix $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, and find a corresponding eigenvector.

The eigenvector must be ________________, but an eigenvalue may be __________.

So λ is an eigenvalue of an $n \times n$ matrix, if and only if

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

What would another name for the solutions to this equation be?

But we already know that any ____________________ is a ____________________ of $\mathbb{R}^n$, so we call it the ___________________________ of *A*.

Ex 4: Find a basis for the eigenspace given $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \lambda = 3$

5.1: Eigenvectors and Eigenvalues

Theorem 1
The eigenvalues of a triangular matrix are the entries on its main diagonal.

Ex 5: Find the eigenvalues of $\begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

What does it mean for a matrix A to have an eigenvalue of 0?

This means that 0 is an eigenvalue of A if and only if A is ________ ______________.

This will be added to our ______________ ______________ ______________ in 5.2.

Theorem 2
If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Proof:

5.1: Eigenvectors and Eigenvalues

Practice Problems

1. Is 5 an eigenvalue of $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?

2. If **x** is an eigenvector of *A* corresponding to λ, what is $A^3\mathbf{x}$?

4. If *A* is an $n \times n$ matrix and λ is an eigenvalue of *A*, show that 2λ is an eigenvalue of 2*A*.

5.2: The Characteristic Equation

Math 220: Linear Algebra

To find eigenvalues of a square matrix, we are finding non-trivial solutions to the equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$. By the invertible matrix theorem, this is the same as finding λ such that $A-\lambda I$ is ______________________________. But this occurs when the ______________________ is ________________.

Ex 1: Find the Eigenvalues of $A=\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

Theorem The Invertible Matrix Theorem (continued)
Let *A* be an $n \times n$ matrix. Then *A* is invertible if and only if:

s. The number 0 is *not* an eigenvalue of *A*.

t. The determinant of *A* is *not* zero.

Theorem 3 Properties of Determinants

Let A and B be $n \times n$ matrices.

a. A is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

We can now determine when the matrix $A-\lambda I$ is not invertible by solving the ______________________ ________________, $\det(A-\lambda I)=0$.

Ex 2: Find the characteristic equation and eigenvalues of $A=\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$.

5.2: The Characteristic Equation

Ex 3: Find the characteristic equation of $A=\begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -1 & 2 & 3 & 0 \\ 5 & 0 & 1 & -1 \end{bmatrix}$.

If A is an $n \times n$ matrix, then $\det(A-\lambda I)$ is a polynomial of ________________ called the ______________________ ______________________ of A.

The eigenvalue of 3 in Ex 3. is said to have ______________________________ because the factor __________ occurs __________ in the characteristic polynomial.

Ex 4: The characteristic polynomial of a 7×7 matrix is $\lambda^7-8\lambda^5+16\lambda^3$. Find the eigenvalues and their multiplicities.

5.2: The Characteristic Equation

Similarity

Two $n \times n$ matrices A and B are considered ________________ if there is an invertible matrix P such that

or

We can also write Q for P^{-1} and get

or

Vocabulary: Changing A into __________ is called the ________________________.

Theorem 4
If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof:

Warnings:

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

Practice Problem

Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

5.3: Diagonalization

Math 220: Linear Algebra

Ex 1: If $D=\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ find D^2, D^3, and D^k.

If $A=PDP^{-1}$ for some invertible *P* and diagonal *D*, then A^k is also easy to compute.

Ex 2: Let $A=\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$. Find a formula for A^k given that $A=PDP^{-1}$, where

$P=\begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}$ and $D=\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

5.3: Diagonalization

A square matrix *A* is said to be __________________________ if *A* is similar to a diagonal matrix *D*.

Theorem 5 The Diagonalization Theorem
An $n \times n$ matrix *A* is diagonalizable if and only if *A* has *n* linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with *D* a diagonal matrix, if and only if the columns of *P* are *n* linearly independent eigenvectors of *A*. In this case, the diagonal entries of *D* are eigenvalues of *A* that correspond, respectively, to the eigenvectors in *P*.

These eigenvectors, since they are linearly independent, form a __________________.

Ex 3: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. That is, find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. The eigenvalues are $\lambda = 1, 5$.

5.3: Diagonalization

Ex 4: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Theorem 6
An $n \times n$ matrix with *n* distinct eigenvalues is diagonalizable.

Note: Having distinct eigenvalues is not a requirement for diagonalizable (see Ex 3).

Theorem 7
Let *A* be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix *A* is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals *n*, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If *A* is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each *k*, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.

5.3: Diagonalization

Ex 5: Diagonalize the matrix, if possible. $A=\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

5.3: Diagonalization

Practice Problems

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

3. Let A be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

5.4-6: Eigenvalues and Dynamical Systems

Math 220: Linear Algebra

Real Eigenvalues

Ex 1: A stretch of desert in Northwestern Mexico is populated mainly by two species of animals: coyotes and roadrunners. We wish to model the populations $c(t)$ and $r(t)$ of coyotes and roadrunners *t* years from now if the current populations c_0 and r_0 are known.

From this habitat, the following equations model the transformation of this system from one year to the next, from time *t* to time *t*+1.:

$$\begin{cases} c(t+1) = 0.86c(t) + 0.08r(t) \\ r(t+1) = -0.12c(t) + 1.14r(t) \end{cases}$$

a.) Write this as a matrix product $\vec{x}(t+1) = A\vec{x}(t)$

We call $\vec{x}(t)$ the ____________________ and $\vec{x}(0)$ the ____________________

This linear transformation is an example of a ________________________

b.) Suppose we begin with 100 coyotes and 300 road runners, find a close-form formula for $c(t)$ and $r(t)$.

5.4-6: Eigenvalues and Dynamical Systems

c.) Suppose we have $c_0 = 200$ and $r_0 = 100$, find $\vec{x}(t)$

d.) Suppose we have $c_0 = r_0 = 1000$, find $\vec{x}(t)$. Hint: Write $\vec{x}_0$ in terms of the eigenbasis.

e.) Sketch a phase portrait to describe this system

5.4-6: Eigenvalues and Dynamical Systems

Here is another example.

Ex 2: Consider $A=\begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$. Since the sum of each column is 1, this linear transformation matrix is called a ______________________.

a.) Find a closed-form expression for A^t. Hint: Since *A* is a transition matrix, one of its eigenvalues will be one.

b.) If $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $A\vec{x}_0$

c.) Find the steady-state or equilibrium vector $\vec{x}_{\text{equ}} = \lim_{t\to\infty} A^t \vec{x}_0$

Complex Eigenvalues

Up to this point, we have only discussed real eigenvalues and real-valued vectors (including eigenvectors). But the linear algebra world we have established works over complex numbers of the form $z = a + bi$ where $i^2 = -1$.

Ex 3: Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^n$ of the matrix $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

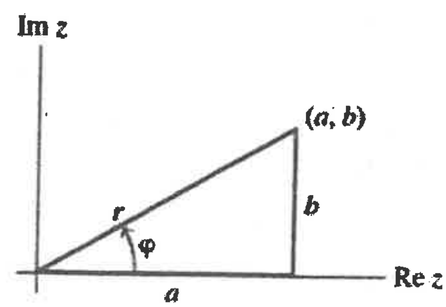
Then write the eigenvectors $\bar{x}$ in the form $\text{Re}\,\bar{x} + i\,\text{Im}\,\bar{x}$

Notice that a real-valued matrix can have complex eigenvalues and eigenvectors. Notice further that the eigenvalues and vectors come in conjugate pairs.

Ex 4: Next we need to unpack the rotation-scaling matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

a.) Find the eigenvalues of *C*.

b.) Let's call $r = |\lambda| = \sqrt{a^2 + b^2}$. Then using the picture below, find $\frac{a}{r}$ and $\frac{b}{r}$ in terms of φ.



So $C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} =$

where is a scaling matrix and is a rotation matrix.

Ex 5: The matrix $\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$ is a rotation-scaling matrix. Find its eigenvalues, scaling factor, and the angle of rotation φ.

5.4-6: Eigenvalues and Dynamical Systems

This brings us back to the idea of matrix factorization. Recall that if *A* had real eigenvalues and enough linearly independent eigenvectors, then $A = PDP^{-1}$ where the columns of *P* were the eigenvectors and *D* was a diagonal matrix whose diagonal entries were the corresponding eigenvalues.

Similarly, let *A* be a real $2x2$ matrix with a complex eigenvalue $\lambda = a - ib \ (b \neq 0)$ and an associated eigenvector $\vec{v}$ in $\mathbb{C}^2$. Then $A = PCP^{-1}$ where $P = [\text{Re}\vec{v} \quad \text{Im}\vec{v}]$ and *C* is the rotation-scaling matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Ex 6: Find an invertible matrix *P* and a matrix *C* of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the matrix $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ has the form $A = PCP^{-1}$

5.4-6: Eigenvalues and Dynamical Systems

Trajectories of Dynamical Systems

When we began this lesson, we used a predator-prey example involving coyotes and road runners. We ended that example with a phase portrait that helped us understand the trajectories based upon various initial state vectors.

Let's begin by trying to understand how these trajectories work.

Ex 7: Suppose $A=\begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix}$ and $\vec{x}_0=\begin{bmatrix} 100 \\ 100 \end{bmatrix}$, find and plot $\vec{x}(1),\vec{x}(2),\vec{x}(3),\ldots,\vec{x}(10)$

5.4-6: Eigenvalues and Dynamical Systems

Ex 7: (revisited) $A=\begin{bmatrix}0.8 & 0\\ 0 & 0.64\end{bmatrix}$ and has eigenvalues $\lambda_1=0.8$ and $\lambda_2=0.64$ with corresponding eigenvectors $\vec{v}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}0\\1\end{bmatrix}$.

So if $\vec{x}_0=\begin{bmatrix}c_1\\c_2\end{bmatrix}=c_1\vec{v}_1+c_2\vec{v}_2$, then $\vec{x}_k=c_1(0.8)^k\begin{bmatrix}1\\0\end{bmatrix}+c_2(0.64)^k\begin{bmatrix}0\\1\end{bmatrix}$

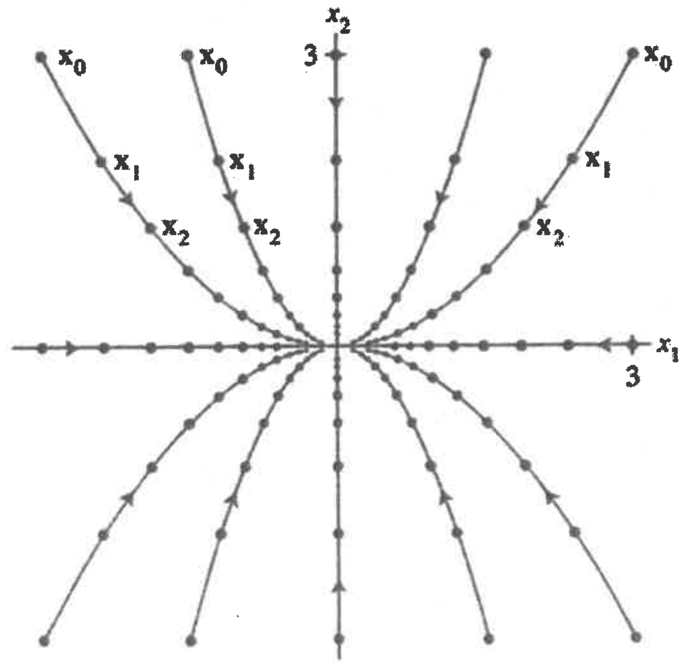


FIGURE 1 The origin as an attractor.

Ex 8: Suppose $A=\begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$. What are the eigenvalues and eigenvectors?

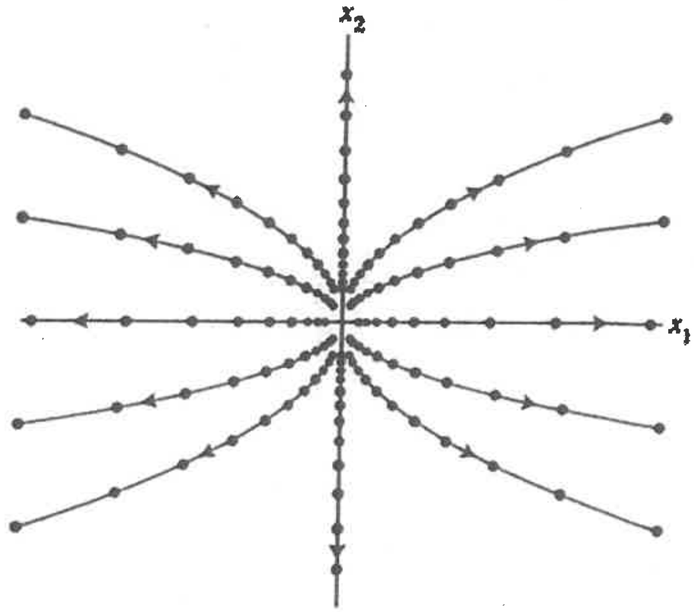


FIGURE 2 The origin as a repeller.

Ex 9: Suppose $A=\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$. Here is a phase portrait for it.

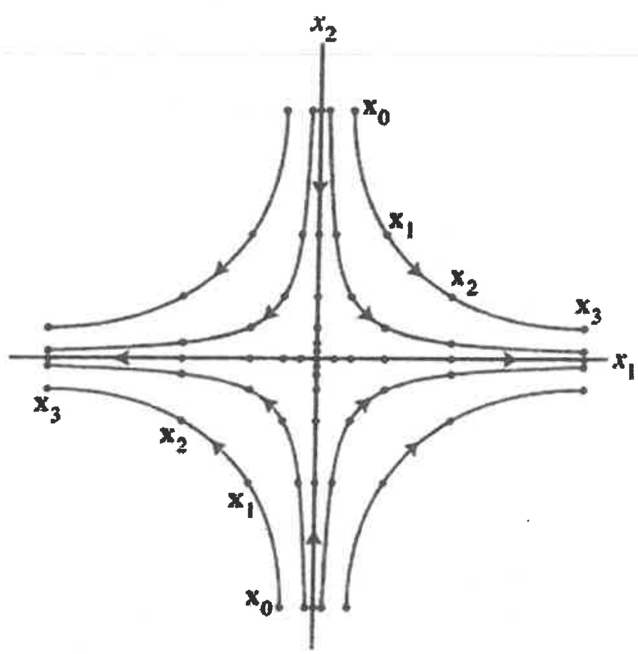


FIGURE 3 The origin as a saddle point.

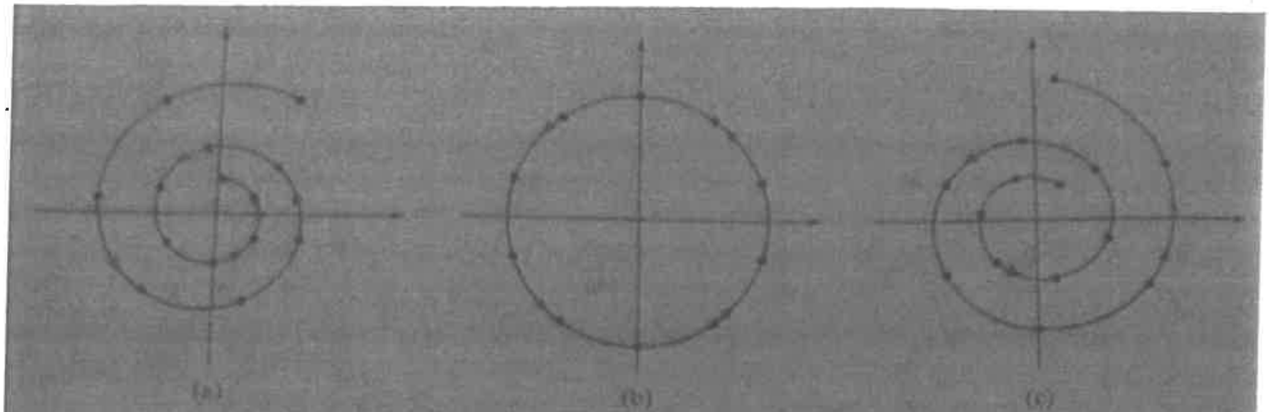
5.4-6: Eigenvalues and Dynamical Systems

Question: In the previous examples, we have focused on diagonal matrices? Is this reasonable? Is it overly simplistic? Explain.

Ex 10: Show that the origin is a saddle point for the solutions of $\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$.

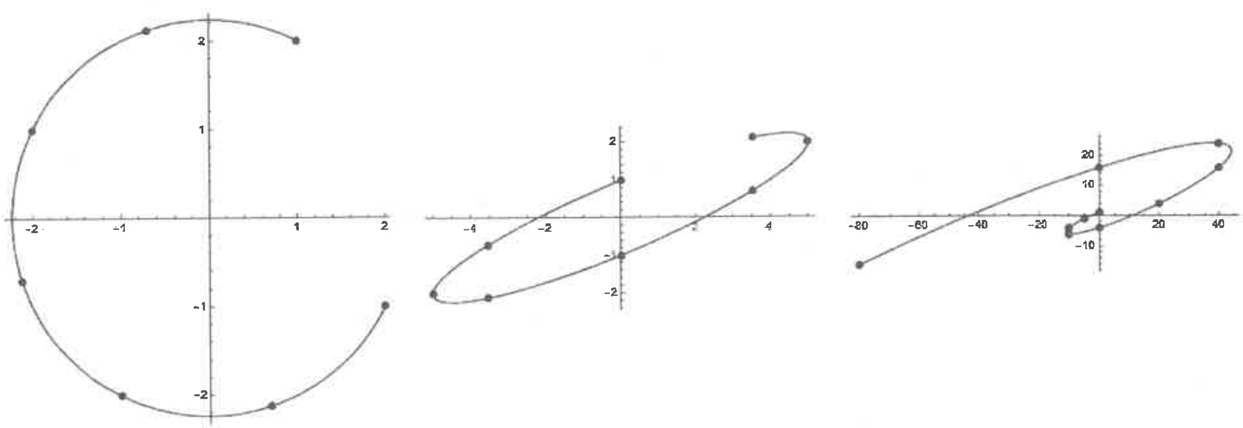
5.4-6: Eigenvalues and Dynamical Systems

Phase portraits get more interesting with complex eigenvalues



(a) $r < 1$: trajectories spiral inward (b) $r = 1$: trajectories are circles
(c) $r > 1$: trajectories spiral outward

Ex 11: Consider the dynamical system and sketch the trajectory of $\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



6.1: Inner Product, Length, and Orthogonality

Math 220: Linear Algebra

If **u** and **v** are vectors in $\mathbb{R}^n$ then we can think of them as $n \times 1$ matrices.

So $\mathbf{u}^T$ is a ____________ matrix and the product of $\mathbf{u}^T\mathbf{v}$ is a ____________ matrix.

We will write this as a real number without brackets, and call $\mathbf{u}^T\mathbf{v}$ the ____________ ____________ of **u** and **v**. It is also written as $\mathbf{u} \cdot \mathbf{v}$ and called the ____________ ____________.

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Ex 1: Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$

Theorem 1

Let **u**, **v**, and **w** be vectors in $\mathbb{R}^n$, and let *c* be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

6.1: Inner Product, Length, and Orthogonality

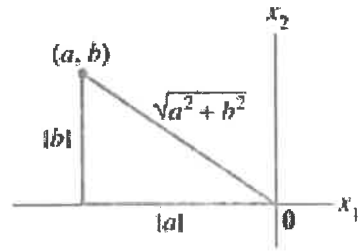
$$(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{w} = c_1(\mathbf{u}_1\cdot\mathbf{w}) + \cdots + c_p(\mathbf{u}_p\cdot\mathbf{w})$$

Definition
The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \text{ and } \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$$

In $\mathbb{R}^2$ this is essentially the

________________________ theorem.



$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

A vector whose length is one is called the ____________ vector.

If we divide a non-zero vector **v** by it's length, ______________________________ we get a unit vector in the same direction as **v**. This is called ______________________.

Ex 2: Let $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ -2 \end{bmatrix}$. Find a unit vector **u** in the same direction as **v**.

Ex 3: Let *W* be a subspace of $\mathbb{R}^2$ spanned by $\mathbf{x} = \begin{bmatrix} 3/4 \\ -2 \end{bmatrix}$. Find a unit vector basis for W.

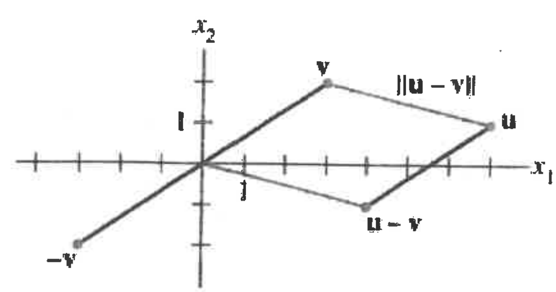
How do we find the distance between two numbers on a number line?

Definition

For **u** and **v** in $\mathbb{R}^n$, the **distance between u and v**, written as $\text{dist}(\mathbf{u},\ \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Ex 4: Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$.



Ex 5: Find the formula for the distance between two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$

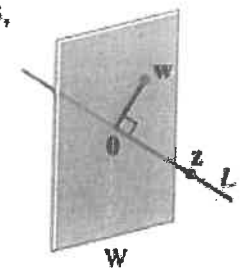
Definition
Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 2 The Pythagorean Theorem
Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

If a vector **z** is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then **z** is said to be ______________________________. The set of all of these orthogonal vectors to W is called the ____________________ _________________of W and is denoted by $W^{\perp}$.

Ex 6: Let *W* be a plane through the origin in $\mathbb{R}^3$, and let *L* be the line through the origin and perpendicular to *W*. If **z** and **w** are nonzero, **z** is on *L*, and **w** is in *W*, then the line segment from **0** to **z** is perpendicular to the line segment from **0** to **w**; that is, $\mathbf{z} \cdot \mathbf{w} = 0$. See Figure 7. So each vector on *L* is orthogonal to every **w** in *W*. In fact, *L* consists of *all* vectors that are orthogonal to the **w**'s in *W*, and *W* consists of all vectors orthogonal to the **z**'s in *L*. That is,

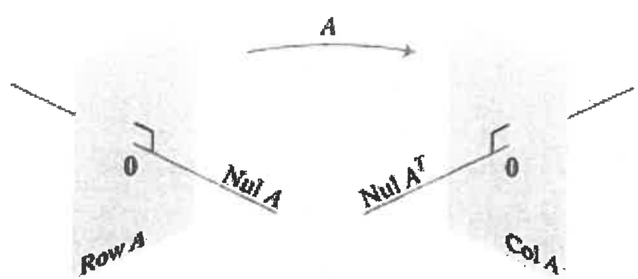
$$L = W^{\perp} \text{ and } W = L^{\perp}$$



1. A vector **x** is in $W^{\perp}$ if and only if **x** is orthogonal to every vector in a set that spans *W*.

2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Remember our comment in 4.6 that the Null Space and Row Space are essentially orthogonal to each other.



Theorem 3

Let *A* be an $m \times n$ matrix. The orthogonal complement of the row space of *A* is the null space of *A*, and the orthogonal complement of the column space of *A* is the null space of A^T:

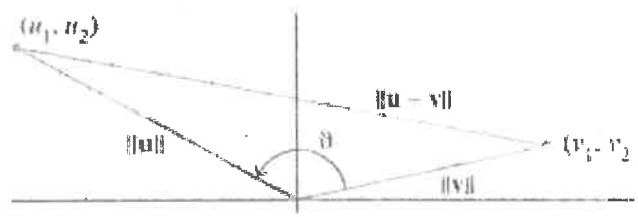
$$(\text{Row } A)^{\perp} = \text{Nul } A \text{ and } (\text{Col } A)^{\perp} = \text{Nul } A^T$$

6.1: Inner Product, Length, and Orthogonality

Ex 7: Using the Null Space and Row Space of **Ex 5 from 4.5**, check that random vectors from each are orthogonal to each other.

Ex 8: Show that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \ \|\mathbf{v}\| \cos v$ where V is the angle between the two vectors, using the Law of Cosines,

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\,\|\mathbf{u}\| \ \|\mathbf{v}\| \cos v$$



6.2: Orthogonal Sets

Math 220: Linear Algebra

A set of vectors $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ is called an ______________ ________ if each pair of distinct vectors from the set is orthogonal. That is, ____________ when $i \neq j$.

Ex 1: Determine whether the set of vectors is orthogonal.

a) $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

Theorem 4
If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof:

6.2: Orthogonal Sets

Definition

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 5

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

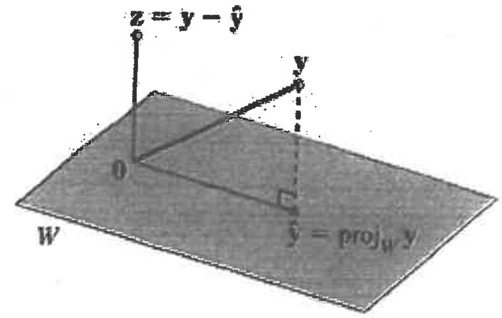
$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \qquad (j = 1, \ldots, p)$$

Ex 2: The vector $\mathbf{v} = \begin{bmatrix} 4 \\ -8 \\ -10 \\ 17 \end{bmatrix}$ is in the subspace W with orthogonal basis from Ex 1b).

Express **v** as a linear combination of the orthogonal basis. $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

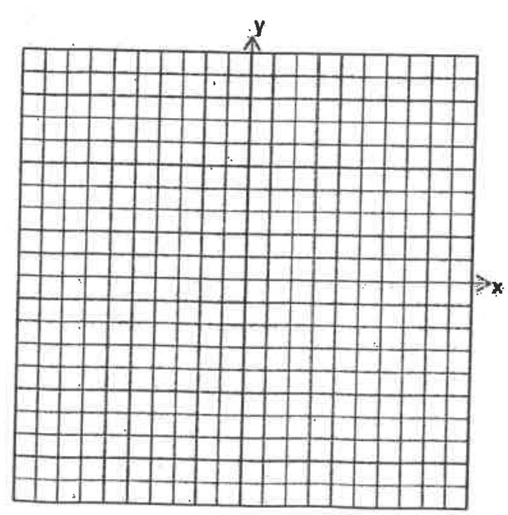
6.2: Orthogonal Sets

An Orthogonal Projection



$$\boxed{\hat{\mathbf{y}} = \operatorname{proj}_L \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}}$$

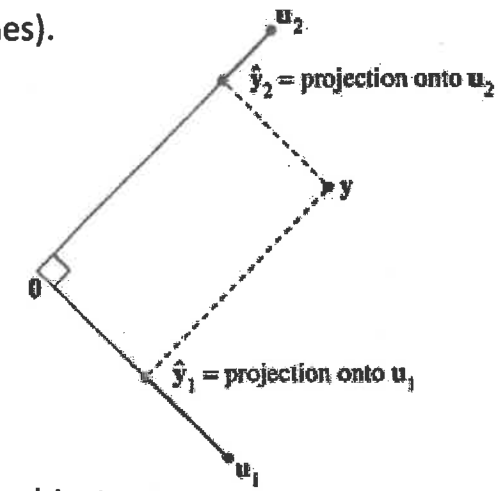
Ex 3: Compute the orthogonal projection of $\begin{bmatrix}1\\7\end{bmatrix}$ onto the line through $\begin{bmatrix}-4\\2\end{bmatrix}$ and the origin. Then write $\begin{bmatrix}1\\7\end{bmatrix}$ as a sum of two orthogonal vectors. Also, observe geometrically.



6.2: Orthogonal Sets

Ex 4: Find the distance from the vector $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ to the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ (from Ex 3).

Notice that the orthogonal projection formula matches the weights of the orthogonal basis terms in theorem 5. Theorem 5 decomposes a vector into a sum of orthogonal projections onto one-dimensional subspaces (lines).

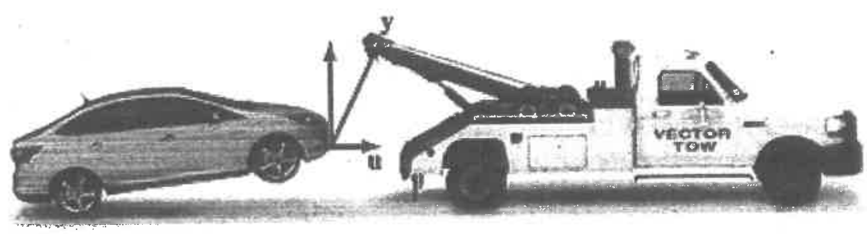


In $\mathbb{R}^2$, if we have an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$

then any $\mathbf{y} \in \mathbb{R}^2$ can be written as

$$\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2$$

In physics we use this to decompose force on an object.



A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called an ______________________ ________ if it is an orthogonal set of ____________ ______________. If W is spanned by this set, then the set is an ___________________________ ______________ for W.

The simplest orthonormal basis for $\mathbb{R}^n$ is $\{ \qquad\qquad\qquad \}$.

Any nonempty subset of this standard basis is orthonormal as well.

Ex 5: Determine whether the set of vectors is orthonormal. Is it an orthonormal basis for $\mathbb{R}^3$? $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Proof:

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

6.2: Orthogonal Sets

Ex 6: Let $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \sqrt{6} \\ 4 \end{bmatrix}$. Verify that $\|U\mathbf{x}\| = \|\mathbf{x}\|$

An ______________________ ______________________ is a square invertible matrix U, such that $U^{-1} = U^T$. By theorem 6, it has orthonormal columns.

The matrix formed from the vectors from Ex 5 is an example.

$$\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 3/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{2} \\ 3/\sqrt{20} & -1/\sqrt{20} & 1/\sqrt{2} \end{bmatrix}$$

Practice Problem

1. Let U and **x** be as in example 6, and let $\mathbf{y} = \begin{bmatrix} -\sqrt{3} \\ \sqrt{2} \end{bmatrix}$. Verify that $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

6.3 & 6.4: Orthogonal Projections & Gram-Schmidt

Math 220: Linear Algebra

Big Picture: We are building to a method (Gram-Schmidt Orthogonalization) that will allow us to use an existing basis to create an orthonormal basis. These concepts will then help us to develop a method for calculating least square models.

Given a vector $\mathbf{y}$ and a subspace W in $\mathbb{R}^n$ there is a vector $\hat{\mathbf{y}} \in W$ such that

1) $\hat{\mathbf{y}}$ is the unique vector in *W* for which $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to *W*

2) $\hat{\mathbf{y}}$ is the unique vector in *W* closest to $\mathbf{y}$

Theorem 8 The Orthogonal Decomposition Theorem
Let *W* be a subspace of $\mathbb{R}^n$. Then each y in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \qquad (1)$$

where $\hat{\mathbf{y}}$ is in *W* and z is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of *W*, then

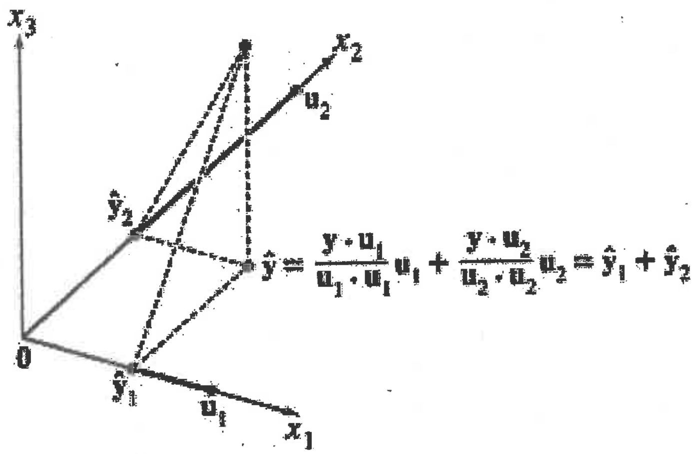
$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \qquad (2)$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Ex 1: Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

6.3 & 6.4: Orthogonal Projections & Gram-Schmidt



Theorem 9 The Best Approximation Theorem
Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \qquad (3)$$

for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$.

Ex 2: As in Ex 1, $\begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$ is the closest point in $W = \text{Span}\left\{\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}\right\}$ to $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Find the distance from $\mathbf{y}$ to W

Practice Problems

1. Let $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Use the fact that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to compute $\text{proj}_W\, \mathbf{y}$.

2. Let *W* be the subspace spanned by the **u**'s, and write **y** as the sum of a vector in *W* and a vector orthogonal to *W*.

$$\mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

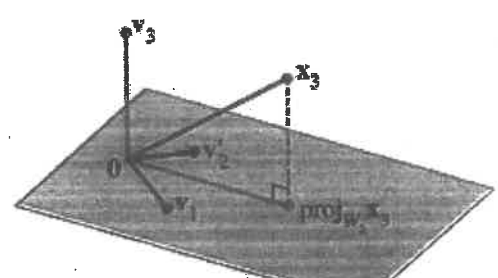
6.3 & 6.4: Orthogonal Projections & Gram-Schmidt

The Gram-Schmidt Process

Ex 3: Let $W = \text{Span}\left\{ \mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \right\}$, construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Ex 4:

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus is a basis for a subspace W of $\mathbb{R}^4$. Construct an orthogonal basis for W.



$W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2'\}$

Theorem 11 The Gram–Schmidt Process
Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p$$

The result of this is that every nonzero subspace W in $\mathbb{R}^n$ has an orthogonal basis.

An orthonormal basis is constructed easily by normalizing all the $\mathbf{v}_k$'s to unit vectors.

Ex 5: Re-write the orthogonal basis found in Ex 3 as an orthonormal basis.

6.3 & 6.4: Orthogonal Projections & Gram-Schmidt

Practice Problems

1. Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

   Construct an orthonormal basis for *W*.

2. Use the Gram–Schmidt process to produce an orthogonal basis for *W*.

$$W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \text{ where } \mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

6.5 & 6.6: Least-Squares Problems
Math 220: Linear Algebra

We will now look at the case where $A\mathbf{x}=\mathbf{b}$ has no solution. What would be "closest" possible solution $\mathbf{x}$? This is called the Least-Squares problem, and it mirrors our Best-Approximation Theorem from 6.3.

Definition
If A is $m \times n$ and **b** is in $\mathbb{R}^m$, **a least-squares solution of** $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

for all **x** in $\mathbb{R}^n$.

Theorem 13
The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$. coincides with the nonempty set of solutions of the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.

6.5 & 6.6: Least-Squares Problems

Ex 1: Find a least-squares solution of the inconsistent system $A\mathbf{x}=\mathbf{b}$ for

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Ex 2: Find a least-squares solution of the inconsistent system $A\mathbf{x}=\mathbf{b}$ for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

6.5 & 6.6: Least-Squares Problems

Theorem 14
Let A be an $m \times n$ matrix. The following statements are logically equivalent:

a. The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each **b** in $\mathbb{R}^m$.

b. The columns of A are linearly independent.

c. The matrix $A^T A$ is invertible.

When these statements are true, the least-squares solution $\hat{\mathbf{x}}$ is given by

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} \qquad (4)$$

The distance from **b** to A**x** is called the ______________________ ____________

Ex 3: Find the least-squares error of Ex 1.

If the columns of A are orthogonal, the least-squares solution is even easier to find.

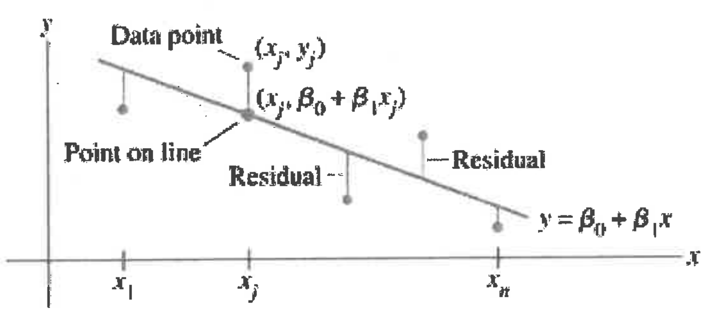
Ex 4: Verify the columns of A are orthogonal and find a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

Practice Problems

1. Let $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$. Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$, and compute the associated least-squares error.

Now we're going to look at finding a best-fit line for a set of data points, also known as linear-regression.



Predicted *y*-value	Observed *y*-value
$\beta_0 + \beta_1 x_1 =$	y_1
$\beta_0 + \beta_1 x_2 =$	y_2
⋮	⋮
$\beta_0 + \beta_1 x_n =$	y_n

$$X\beta = \mathbf{y}, \quad \text{where } X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Ex 5: Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the data points. $(1,1),(4,2),(8,4),(11,5)$

Ex 6: Find the quadratic regression equation $y = \beta_0 + \beta_1 x + \beta_2 x^2$ of the least-squares line that best fits the data points. $(-2,12),(-1,5),(0,3),(1,2),(2,4)$.

6.5 & 6.6: Least-Squares Problems

The General Linear Model

In some applications, it is necessary to fit data points with something other than a straight line. In the examples that follow, the matrix equation is still $X\beta = \mathbf{y}$, but the specific form of X changes from one problem to the next. Statisticians usually introduce a **residual vector** $\in$, defined by $\in = \mathbf{y} - X\beta$, and write

$$\mathbf{y} = X\beta + \in$$

Any equation of this form is referred to as a **linear model**. Once X and $\mathbf{y}$ are determined, the goal is to minimize the length of $\in$, which amounts to finding a least-squares solution of $X\beta = \mathbf{y}$. In each case, the least-squares solution $\hat{\beta}$ is a solution of the normal equations

$$X^T X\beta = X^T \mathbf{y}$$

Ex 7: A certain experiment produces the data (1, 7.9), (2, 5.4), and (3, -0.9). Describe the model that produces a least-squares fit of these points by a function of the form $y = A\cos x + B\sin x$