Math 220
Winter 2024
Assessment 6
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Name: Key

We have to push the boundaries, take the risks. Without that, there is no science. No medicine.
Dr. Michael Morbius in *Morbius* (2022)

No work = no credit

1. Warm-ups

   (a) (1 point) $\mathbb{P}_n$ isomorphic with $\mathbb{R}^{\boxed{n+1}}$

   (b) (1 point) min rank of $A_{2\times 3}$ 0

   (c) (1 point) max rank of $A_{2\times 3}$ 2

2. (1 point) In reference to the quote above, what is required for science and medicine to exist? Answer using complete English sentences.

   We must push boundaries and drink blood if science is to exist. (Jk).

3. (4 points) Prove (or disprove) the Unique Representation Theorem.

   Claim: Let $B = \left\{\vec{b}_1, ..., \vec{b}_n\right\}$ be a basis for a vector space V. Then for each $\vec{x} \in V$, there exists a unique set of scalars $c_1, ..., c_n$ such that $\vec{x} = c_1\vec{b}_1 + ... + c_n\vec{b}_n$.

proof.

Let B and $\vec{x}$ be given as above.

Suppose there are two representations for $\vec{x}$ in terms of basis B.

$\Rightarrow \vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ and $\vec{x} = d_1\vec{b}_1 + \dots + d_n\vec{b}_n$ for scalars $c_1, \dots, c_n$ and $d_1, \dots, d_n$.

$\Rightarrow \vec{0} = \vec{x} - \vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n - d_1\vec{b}_1 - \dots - d_n\vec{b}_n$

$= (c_1 - d_1)\vec{b}_1 + \dots + (c_n - d_n)\vec{b}_n$

This is a homogeneous equation. Since B is L.I. (since it's a basis), the homogeneous equation has only the trivial soln: $c_1 - d_1 = 0 \dots c_n - d_n = 0$

$\Rightarrow c_1 = d_1, \dots, c_n = d_n$

$\therefore$ The representation for $\vec{x}$ is unique.

4. (4 points) Consider the subspace $H = \{(a,b,c) : a - 3b + c = 0, b - 2c = 0, \cancel{2b - c = 0}\}$. Find a basis and state the dimension of H.

$a - 2b - c = 0$

H is the set of solutions to:

$$\overset{A}{\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & -1 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \overset{\vec{0}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$\Rightarrow H = \text{Nul } A$

basis for H

$$A \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Nul } A = \text{span}\left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \right\}$$

and $\dim(H) = 1$

5. (4 points) Find $[\vec{x}]_{\mathcal{B}}$ if $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$ and $\vec{x} = \begin{bmatrix} 13 \\ 4 \end{bmatrix}$

$$\underset{E \leftarrow B}{P} = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$$

$$[\vec{x}]_B = \underset{E \leftarrow B}{P^{-1}} \vec{x}$$

$$= \frac{1}{25} \begin{bmatrix} 8 & -1 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} 13 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

6. (2 points) True or False: The number of variables in the equation $A\vec{x} = \vec{0}$ equals the nullity of A. Justify your answer.

False. The number of <u>free</u> variables.

7. (4 points) Let $\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and find the $\mathcal{C}$-coordinates of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}}$

Find $\underset{C \leftarrow B}{P}$

$$\left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{c|cc} I & 8 & 3 \\ & -5 & -2 \end{array}\right]$$

$$[\vec{x}]_C = \underset{C \leftarrow B}{P} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B$$

$$= \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -7 \end{bmatrix}$$

Check that both have the same $\vec{x}$,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_B = 1\begin{bmatrix} 7 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -3 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 11 \\ -7 \end{bmatrix}_C = 11\begin{bmatrix} 4 \\ 1 \end{bmatrix} - 7\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -3 \end{bmatrix} \checkmark$$