Assessment 6 (10 or 11 am)
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Math 220

Name (first & last): Key

Either mathematics is too big for the human mind or the human mind is more than a machine.

Kurt Gödel
1906-1978 (Austrian mathematician)

No work = no credit

Warm-ups (1 pt each): $\vec{e}_2 + \vec{e}_1^{\,T} =$ undefined mismatched dim $\quad \vec{e}_2\vec{e}_1^{\,T} = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \quad -3^2 = -9$

1.) (1 pt) Based upon the quote by Gödel (above), did Gödel believe that all mathematics could be performed by a machine? Please answer using complete sentences.

Gödel thought math is - more than machines.

2.) (4 pts) Show that $B = \{1+t, 1+t^2, t+t^2\}$ is a basis for P_2 and then find the coordinate vector of $p(t) = 6+3t-t^2$ relative to B.

$$B \longleftrightarrow \left\{ \overset{\vec{b}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \overset{\vec{b}_2}{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \overset{\vec{b}_3}{\begin{bmatrix}0\\1\\1\end{bmatrix}} \right\}$$

And $P_B = \begin{bmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{bmatrix} \sim I$ so the vectors are L.I. and form a basis.

$$\begin{tikzcd} \vec{x} \arrow[r, bend right, "P_B^{-1}"'] & {[\vec{x}]_B} \arrow[l, bend right, "P_B"'] \end{tikzcd}$$

$$P_B^{-1}\begin{bmatrix}6\\3\\-1\end{bmatrix} = \begin{bmatrix}5\\1\\-2\end{bmatrix} = [\vec{p}]_B$$

or $p(t) = 5(1+t) + 1(1+t^2) - 2(t+t^2)$

$\overline{8}$

$\overline{23}$

3.) (8 pts) Given the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 4 \\ 1 & 2 & 4 & -1 \end{bmatrix}$, answer the following.

a) Find a basis for the column space of A.

$$A \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for col $A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

b) Find Nul A

nul $A =$ span $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

c) What is the dimension of the null space of A and what is the rank of A?

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d) The zero vector is in both the null space and column space. List all vectors (if any) in the intersection (overlap) of the two subspaces.

col $A \subset \mathbb{R}^3$ the intersection

nul $A \subset \mathbb{R}^4$ is the empty set.

$\overline{8}$

4.) (4 pts) Complete the proof of the Unique Representation Theorem

Claim: Let $B = \{\vec{b}_1, ..., \vec{b}_n\}$ be a basis for a vector space V. Then for each $\vec{x}$ in V, there exists a unique set of scalars $c_1, ... c_n$ such that $\vec{x} = c_1\vec{b}_1 + ... + c_n\vec{b}_n$.

Proof.

Let $\vec{x} \in V$ be given. Assume there are two representations for $\vec{x}$ with respect to $\mathcal{B}$.

so $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ and

$\vec{x} = d_1\vec{b}_1 + \dots + d_n\vec{b}_n$ for scalars $c_1, \dots, c_n$ and $d_1, \dots, d_n$

$\Rightarrow \vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + \dots + (c_n - d_n)\vec{b}_n$

but $\vec{b}_1, \dots, \vec{b}_n$ is a basis so there are no non-trivial solns. to the homogeneous eqn.

$\Rightarrow c_i - d_i = 0$ or $c_i = d_i$ for $i = 1, \dots, n$

$\therefore \vec{x}$ has a unique representation.

5.) (3 pts) True or False (circle one). Justify your answer.

a.) (T or F) If $H = \text{Span}\{\vec{b}_1, ..., \vec{b}_p\}$, then $\{\vec{b}_1, ..., \vec{b}_p\}$ is a basis for H?

False. $\vec{b}_1, \dots, \vec{b}_p$ must also be LI to be a basis.

b.) (T or F) The dimension of the vector space P_4 is 4.

False. $\dim(\mathbb{P}_4) = 5$

c.) (T or F) The dimensions of the row space and the column space of A are the same, even if A is not square.

True. $\dim(\text{row } A) = \dim(\text{col } A) = \#$ of pivots in A.

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