100	90's	80's	70's	60's	<60
0	3	5	2	5	6

21 min.

Math 220
Winter 2024
Assessment 3
Dusty Wilson

high = 95%

$\bar{x}$ = 71%

med = 68.3%

Name: Key

[S]ooner or later you're going to realize just as I did that there's a difference between knowing the path and walking the path.
Morpheus in *The Matrix* (1999)

No work = no credit

1. Warm-ups

   (a) (1 point) $\vec{e}_1$ $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   (b) (1 point) $\vec{e}_2\,\vec{e}_1^{\,T}$ $= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

   (c) (1 point) $\vec{e}_1^{\,T}\vec{e}_2$ $= \begin{bmatrix} 0 \end{bmatrix}$

2. (1 point) In light of the quote by Morpheus, what is a path that you know and yet struggle to walk? Answer using complete English sentences.

   I know the path of listening and empathy, but I always feel like I am back @ step 1.

3. (4 points) What does it mean for a set to be linearly independent?

   No element of the set is a linear combination of the other elements of the set.

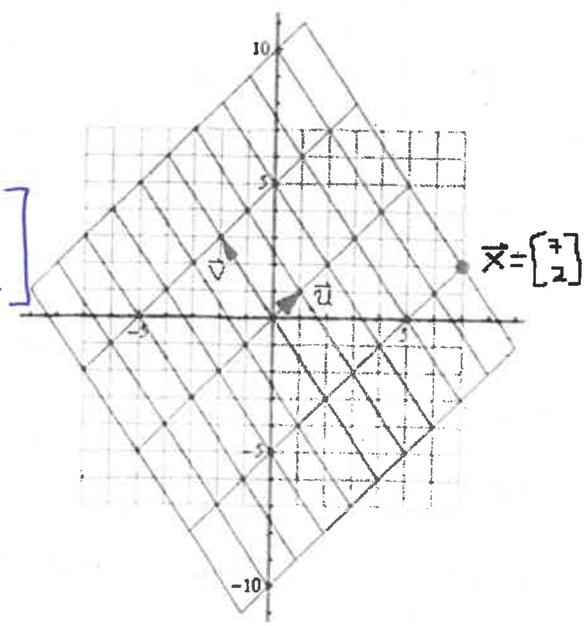
4. (4 points) (a.) Suppose T is a linear transformation such that $T(\vec{e}_1) = \vec{u}$ and $T(\vec{e}_2) = \vec{v}$.

   (a.) Find the matrix A of the linear transformation where $T(\vec{x}) = A\vec{x}$

   (b.) find $T(\begin{bmatrix} 5 \\ -1 \end{bmatrix})$.

(a) $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

(b) $T(\begin{bmatrix} 5 \\ -1 \end{bmatrix}) = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$



5. (2 points) True or False: If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S. Justify your answer.

False. For example $\left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}0\\2\end{bmatrix}}_{\vec{v}_3} \right\}$. Notice that $\vec{v}_1$ is not a linear combination of $\vec{v}_2, \vec{v}_3$.

6. (4 points) Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a pivot in every column, the columns of A form a L.I. set.

7. (4 points) Consider the linear transformation T where $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

a.) Describe the "image" of T geometrically.

b.) Is T "onto"? Why or why not?

$$T: \begin{bmatrix}1\\0\end{bmatrix} \longmapsto \begin{bmatrix}1\\1\end{bmatrix}$$

$$T: \begin{bmatrix}0\\1\end{bmatrix} \longmapsto \begin{bmatrix}1\\1\end{bmatrix}$$

(a) The image of T is the line $x_1 = x_2$ ($y=x$). Aka.

(b) T is <u>not</u> <u>onto</u> because there are vectors in the codomain ($\mathbb{R}^2$) that are not in the range (the line $y=x$).

8. (4 points) Prove the following.

Claim: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

proof.

Let $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^N$ w/ $p > N$ be given.

Define $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & & | \end{bmatrix}_{N \times p}$. has more columns than rows.

When we row reduce, there will be at most N pivots (one per row).

Since there are more columns than rows there must be at least one column w/o a pivot.

$\therefore$ The columns of A, $\{\vec{v}_1, \dots, \vec{v}_p\}$ for a linearly dependent set.

Q.E.D.

9. (4 points) (a.) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ that projects objects in 3D onto the x_1 x_2-plane and rotates them ~~30~~ 60 deg counter-clockwise. For example $T(\vec{e}_1) = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$ and $T(\vec{e}_3) = \vec{0}$.

Hint: What is $T(\vec{e}_2)$ $= \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$

a.) What is the matrix of the linear transformation?

b.) Is T "one-to-one"? Why or why not?

(a) $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$

(b) T is not 1-1 because (for exampl),

$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right) = \vec{0}$