6.2: Orthogonal Sets
Math 220: Linear Algebra

A set of vectors $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ is called an orthogonal set if each pair of distinct vectors from the set is orthogonal. That is, $\vec{u}_i \cdot \vec{u}_j = 0$ when $i \neq j$.

Ex 1: Determine whether the set of vectors is orthogonal.

a) $\begin{array}{ccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, & \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, & \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \end{array}$

$\vec{u}_1 \cdot \vec{u}_2 = -12 + 21 - 9 = 0$ ✓

$\vec{u}_1 \cdot \vec{u}_3 = 6 - 7 + 1 = 0$ ✓

$\vec{u}_2 \cdot \vec{u}_3 = -18 - 3 - 9 = -30 \neq 0$.

This is $\boxed{\text{not}}$ an orthogonal set.

b) $\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, & \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, & \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \end{array}$

$\vec{v}_1 \cdot \vec{v}_2 = -3 - 6 - 3 + 12 = 0$

$\vec{v}_1 \cdot \vec{v}_3 = 9 - 16 + 7 + 0 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = -3 + 24 - 21 + 0 = 0$

This is an orthogonal set.

Theorem 4
If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof:

solve $\vec{0} = c_1\vec{u}_1 + \ldots + c_p\vec{u}_p$

consider $\vec{u}_i \cdot \vec{0} = \vec{u}_i \cdot (c_1\vec{u}_1 + \ldots + c_p\vec{u}_p)$ for $i = 1, \ldots, p$

$\Rightarrow 0 = c_i \underbrace{\vec{u}_i \cdot \vec{u}_i}_{\neq 0}$

$\Rightarrow 0 = c_i$ for $i = 1, \ldots, p$.

$\Rightarrow \vec{u}_1, \ldots, \vec{u}_p$ are linearly independent. and clearly the span the subspace.

$\therefore S$ is a basis.

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Definition
An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 5
Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \qquad (j = 1, \ldots, p)$$

To see this, dot both sides with $\vec{u}_j$

$\vec{y}\cdot\vec{u}_j = c_1\vec{u}_1\cdot\vec{u}_j + \cdots c_p\vec{u}_p\cdot\vec{u}_j$

$= 0 + \ldots + c_j\vec{u}_j\cdot\vec{u}_j + 0 + \ldots + 0$

Ex 2: The vector $\mathbf{v} = \begin{bmatrix} 4 \\ -8 \\ -10 \\ 17 \end{bmatrix}$ is in the subspace W with orthogonal basis from Ex 1b).

$$\begin{array}{ccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, & \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, & \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \end{array}$$

Express **v** as a linear combination of the orthogonal basis.

Find $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$

$$c_1 = \frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{12+16-10+51}{9+4+1+9} = \frac{69}{23} = 3$$

$$c_2 = \frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{-4-24+30+68}{1+9+9+16} = 2$$

$$c_3 = \frac{\vec{v}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{12-64-70+0}{9+64+49+0} = -1$$

$\Rightarrow \vec{v} = 3\vec{u}_1 + 2\vec{u}_2 - \vec{u}_3.$

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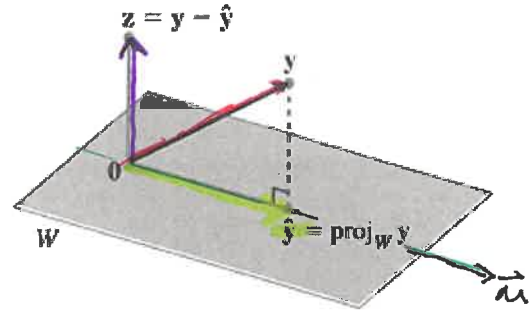
vector $\vec{z}$ is $\perp$ to the space W.

An Orthogonal Projection



vector $\vec{u}$ is on W.

$$\vec{y} = \underbrace{\hat{y}}_{\text{in } W} + \underbrace{\vec{z}}_{\perp \text{ to } W}$$

$$\Rightarrow \vec{y} = \alpha\vec{u} + \vec{z}$$

$$\Rightarrow \vec{z} = \vec{y} - \alpha\vec{u}$$

$$\Rightarrow \vec{z}\cdot\vec{u} = (\vec{y} - \alpha\vec{u})\cdot\vec{u}$$

$$\Rightarrow 0 = \vec{y}\cdot\vec{u} - \alpha\,\vec{u}\cdot\vec{u}$$

$$\Rightarrow \alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} \quad \text{Thus} \quad \hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \quad \text{AND} \quad \vec{z} = \vec{y} - \hat{y}$$

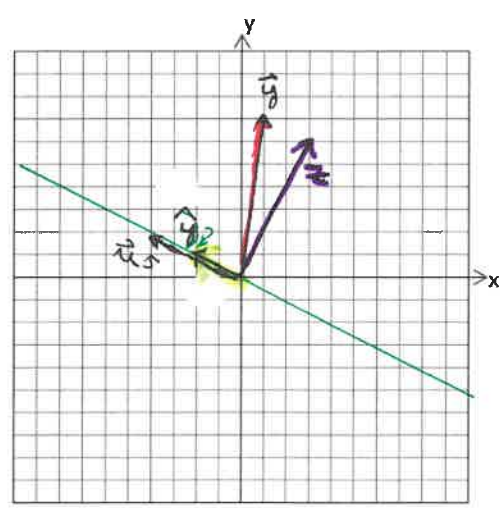
$$\boxed{\hat{\mathbf{y}} = \text{proj}_L\,\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\vec{u}}$$

Ex 3: Compute the orthogonal projection of $\overset{\vec{y}}{\begin{bmatrix}1\\7\end{bmatrix}}$ onto the line through $\overset{\vec{u}}{\begin{bmatrix}-4\\2\end{bmatrix}}$ and the origin.

Then write $\begin{bmatrix}1\\7\end{bmatrix}$ as a sum of two orthogonal vectors. Also, observe geometrically.

$$\hat{y} = \frac{-4+14}{16+4}\vec{u} = \frac{1}{2}\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}1\\7\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$$



Ex 4: Find the distance from the vector $\underset{\vec{y}}{\begin{bmatrix}1\\7\end{bmatrix}}$ to the line through $\underset{\vec{u}}{\begin{bmatrix}-4\\2\end{bmatrix}}$ (from Ex 3).

$\hat{y}$ is the closest point on the line through $\vec{u}$ to $\vec{y}$.

That is, we need $\|\vec{z}\|$

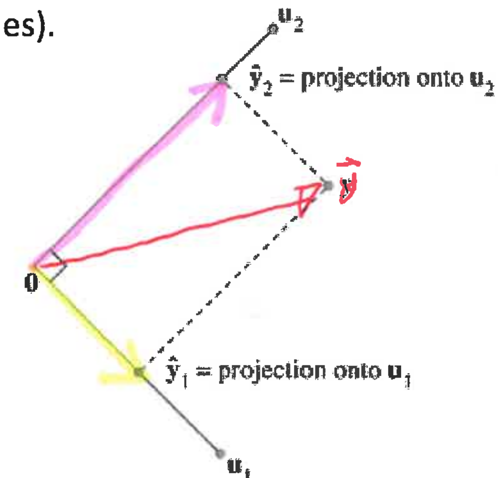
$$\|\vec{z}\| = \|\vec{y}-\hat{y}\|$$
$$= \sqrt{9+36}$$
$$= \sqrt{45}$$

Notice that the orthogonal projection formula matches the weights of the orthogonal basis terms in theorem 5. Theorem 5 decomposes a vector into a sum of orthogonal projections onto one-dimensional subspaces (lines).

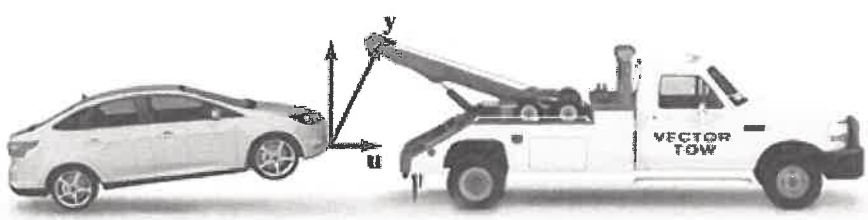
In $\mathbb{R}^2$, if we have an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$

then any $\mathbf{y} \in \mathbb{R}^2$ can be written as

$$\vec{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2$$



In Physics we use this to decompose force on an object.



A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called an orthonormal set if it is an orthogonal set of unit vectors. If *W* is spanned by this set, then the set is an orthonormal basis for *W*.

The Simplest orthonormal basis for $\mathbb{R}^n$ is $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

Any nonempty subset of this standard basis is orthonormal as well.

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Ex 5: Determine whether the set of vectors is orthonormal. Is it an orthonormal basis for $\mathbb{R}^3$?

$$\begin{array}{ccccc} \vec{u}_1 & & \vec{u}_2 & & \vec{u}_3 \\ \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix} & , & \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix} & , & \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \end{array}$$

verify the vectors are unit vectors.

$$\|\vec{u}_1\|^2 = \vec{u}_1 \cdot \vec{u}_1 = \frac{1}{10} + \frac{9}{20} + \frac{9}{20} = 1$$

$$\vec{u}_2 \cdot \vec{u}_2 = \frac{9}{10} + \frac{1}{20} + \frac{1}{20} = 1$$

$$\vec{u}_3 \cdot \vec{u}_3 = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

They are unit vectors.

verify the vectors are orthonormal

$$\vec{u}_1 \cdot \vec{u}_2 = \frac{3}{10} - \frac{3}{20} - \frac{3}{20} = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = 0 - \frac{3}{\sqrt{40}} + \frac{3}{\sqrt{40}} = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = 0 + \frac{1}{\sqrt{40}} - \frac{1}{\sqrt{40}} = 0$$

They are orthogonal

$\therefore$ they form an orthonormal basis for $\mathbb{R}^3$.

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Proof:

Let $U = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix}$ where $\vec{u}_1, \dots, \vec{u}_n$ are orthonormal.

$$\Rightarrow U^T U = \begin{bmatrix} - \vec{u}_1^T - \\ \vdots \\ - \vec{u}_n^T - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec{u}_1^T \vec{u}_1 & \vec{u}_1^T \vec{u}_2 & \cdots & \vec{u}_1^T \vec{u}_n \\ \vec{u}_2^T \vec{u}_1 & \vec{u}_2^T \vec{u}_2 & \cdots & \vec{u}_2^T \vec{u}_n \\ \vdots & \vdots & & \vdots \\ \vec{u}_n^T \vec{u}_1 & \vec{u}_n^T \vec{u}_2 & \cdots & \vec{u}_n^T \vec{u}_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Q.E.D.

Theorem 7

Let U be an $m \times n$ matrix with $\underbrace{\text{orthonormal}}_{\text{orthogonal and unit vectors.}}$ columns, and let x and y be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

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Ex 6: Let $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \sqrt{6} \\ 4 \end{bmatrix}$. Verify that $\underbrace{\|U\mathbf{x}\|}_{LHS} = \underbrace{\|\mathbf{x}\|}_{RHS}$

main point: length is preserved under an orthonormal transformation.

LHS

$$U\vec{x} = \begin{bmatrix} \sqrt{\frac{6}{3}} - \frac{4}{\sqrt{2}} \\ \sqrt{\frac{6}{3}} \\ \sqrt{\frac{6}{3}} + \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 - 2\sqrt{2} \\ \sqrt{2} \\ 2 + 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 3\sqrt{2} \end{bmatrix} \Rightarrow \underbrace{\|U\vec{x}\|}_{\text{vector in } \mathbb{R}^3} = \sqrt{2+2+18} = \sqrt{22}$$

$$\underbrace{\|\vec{x}\|}_{\text{vector in } \mathbb{R}^2} = \sqrt{6+16} = \sqrt{22} \leftarrow \text{The norms are equal.}$$

An <u>orthogonal</u> <u>matrix</u> is a square invertible matrix U, such that $U^{-1} = U^T$. By theorem 6, it has orthonormal columns.

The matrix formed from the vectors from Ex 5 is an example.

$$\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 3/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{2} \\ 3/\sqrt{20} & -1/\sqrt{20} & 1/\sqrt{2} \end{bmatrix}$$

Practice Problem

1. Let U and **x** be as in example 6, and let $\mathbf{y} = \begin{bmatrix} -\sqrt{3} \\ \sqrt{2} \end{bmatrix}$. Verify that $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

$$U\vec{y} = \begin{bmatrix} -1 - 1 \\ -1 + 0 \\ -1 + 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \Rightarrow (U\vec{x}) \cdot (U\vec{y}) = 2\sqrt{2} - \sqrt{2} + 0 = \sqrt{2}$$

AND $\vec{x} \cdot \vec{y} = -3\sqrt{2} + 4\sqrt{2} = \sqrt{2} \leftarrow$ the inner products are equal

main point: angle is preserved under an orthonormal transformation.