6.1: Inner Product, Length, and Orthogonality

Math 220: Linear Algebra

If **u** and **v** are vectors in $\mathbb{R}^n$ then we can think of them as $n \times 1$ matrices.

So $\mathbf{u}^T$ is a ___1 × n___ matrix and the product of $\mathbf{u}^T\mathbf{v}$ is a ___1 × 1___ matrix.

We will write this as a real number without brackets, and call $\mathbf{u}^T\mathbf{v}$ the ___inner product___ of **u** and **v**. It is also written as $\mathbf{u} \cdot \mathbf{v}$ and called the ___dot product___.

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Ex 1: Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} = -10$$

$$\vec{v} \cdot \vec{u} = \vec{v}^T \vec{u} = \begin{bmatrix} -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = -10$$

Theorem 1
Let **u**, **v**, and **w** be vectors in $\mathbb{R}^n$, and let *c* be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

scalar in $\mathbb{R}$ vector in $\mathbb{R}^n$

6.1: Inner Product, Length, and Orthogonality

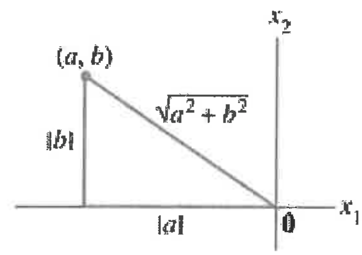
$$\boxed{(c_1\mathbf{u}_1+\cdots+c_p\mathbf{u}_p)\cdot\mathbf{w}=c_1(\mathbf{u}_1\cdot\mathbf{w})+\cdots+c_p(\mathbf{u}_p\cdot\mathbf{w})}$$

Definition
The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\|=\sqrt{\mathbf{v}\cdot\mathbf{v}}=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}, \quad \text{and} \quad \boxed{\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}}$$

In $\mathbb{R}^2$ this is essentially the

Pythagorean theorem.



scalar and norm $\quad \boxed{\|c\mathbf{v}\|=|c|\,\|\mathbf{v}\|}$

A vector whose length is one is called a unit vector.

If we divide a non-zero vector **v** by it's length, that is multiply by $\frac{1}{\|\vec{v}\|}$ we get a unit vector in the same direction as **v**. This is called normalizing.

$\frac{\vec{v}}{\|\vec{v}\|}=\vec{u}$ is a unit vector in the same direction as $\vec{v}$.

Ex 2: Let $\mathbf{v}=\begin{bmatrix}5\\2\\4\\-2\end{bmatrix}$. Find a unit vector **u** in the same direction as **v.**

$\|\vec{v}\|=\sqrt{25+4+16+4}=\sqrt{49}=7$

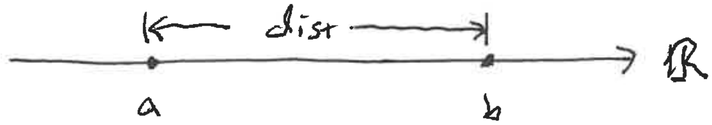
and $\vec{u}=\begin{bmatrix}5/7\\2/7\\4/7\\-2/7\end{bmatrix}$ is a unit vector in the same direction as $\vec{v}$.

Ex 3: Let W be a subspace of $\mathbb{R}^2$ spanned by $\mathbf{x} = \begin{bmatrix} 3/4 \\ -2 \end{bmatrix}$. Find a unit vector basis for W.

$\|\vec{x}\| = \sqrt{\frac{9}{16} + 4} = \sqrt{\frac{73}{16}}$

and $\vec{u} = \frac{4}{\sqrt{73}} \begin{bmatrix} 3/4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{73} \\ -8/\sqrt{73} \end{bmatrix}$ is a unit vector basis for W.

How do we find the distance between two numbers on a number line?



distance $= |b-a|$
$= |a-b|$

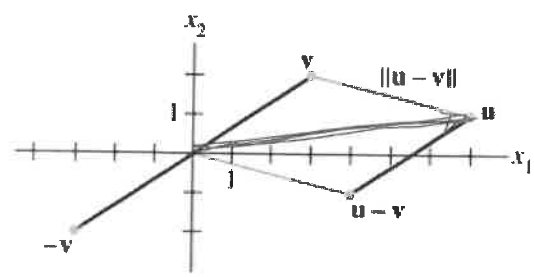
Definition
For **u** and **v** in $\mathbb{R}^n$, **the distance between u and v**, written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Ex 4: Compute the distance between the vectors $\mathbf{u} = (7,1)$ and $\mathbf{v} = (3,2)$.

$\vec{u} - \vec{v} = (7,1) - (3,2) = (4,-1)$

$\Rightarrow \text{dist}(\vec{u}, \vec{v}) = \|4,-1\| = \sqrt{16+1} = \sqrt{17}$



6.1: Inner Product, Length, and Orthogonality

Ex 5: Find the formula for the distance between two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$

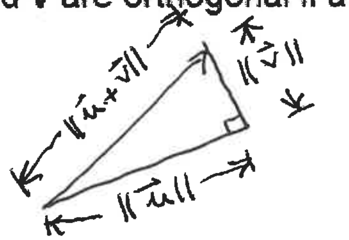
$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

This is the distance formula in $\mathbb{R}^3$.

perpendicular $\perp$

Definition
Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

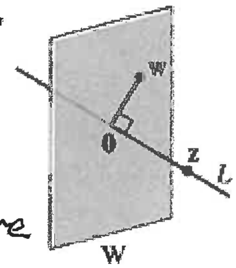
Theorem 2 The Pythagorean Theorem
Two vectors **u** and **v** are orthogonal if and only if $\underbrace{\|\mathbf{u} + \mathbf{v}\|^2}_{c^2} = \underbrace{\|\mathbf{u}\|^2}_{A^2} + \underbrace{\|\mathbf{v}\|^2}_{B^2}$.



If a vector **z** is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then **z** is said to be <u>orthogonal to W</u>. The set of all of these orthogonal vectors to W is called the <u>orthogonal</u> <u>compliment</u> of W and is denoted by $W^{\perp}$.

This gets called "W perp" for short.

Ex 6: Let W be a plane through the origin in $\mathbb{R}^3$, and let L be the line through the origin and perpendicular to W. If **z** and **w** are nonzero, **z** is on L, and **w** is in W, then the line segment from **0** to **z** is perpendicular to the line segment from **0** to **w**; that is, $\mathbf{z} \cdot \mathbf{w} = 0$. See Figure 7. So each vector on L is orthogonal to every **w** in W. In fact, L consists of *all* vectors that are orthogonal to the **w**'s in W, and W consists of all vectors orthogonal to the **z**'s in L. That is,



$$L = W^{\perp} \text{ and } W = L^{\perp}$$

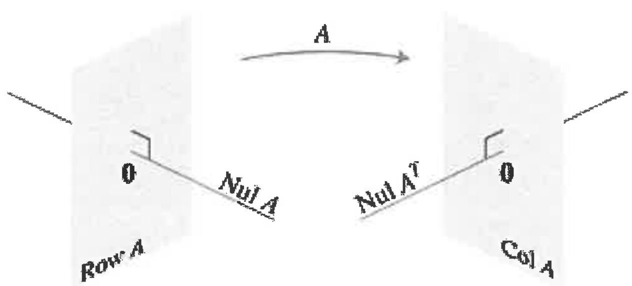
all vectors on the line L are in the space perpendicular to the plane W.

all vectors on the plane W are in the space perpendicular to the line L.

1. A vector **x** is in $W^{\perp}$ if and only if **x** is orthogonal to every vector in a set that spans W.

2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Remember our comment in 4.6 that the Null Space and Row Space are essentially orthogonal to each other.



Theorem 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^{\perp} = \text{Nul } A \text{ and } (\text{Col } A)^{\perp} = \text{Nul } A^T$$

Ex 7: Using the Null Space and Row Space of **Ex 5 from 4.6**, check that random vectors from each are orthogonal to each other.

$$\vec r = \vec r_1 + \vec r_2 + \vec r_3 = \begin{bmatrix} 1 \\ 1 \\ -7 \\ 1 \\ -3 \end{bmatrix}$$

and

$$\vec n = \vec n_1 + \vec n_2 = \begin{bmatrix} -1 \\ 9 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

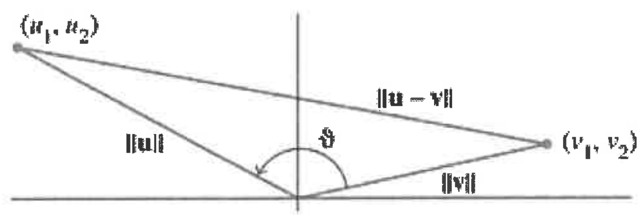
thus $\vec r \cdot \vec n = -1 + 9 - 7 + 2 - 3 = 0$

This example helps us understand what is meant by:

Row $A \perp$ Nul A

Ex 8: Show that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \ \|\mathbf{v}\| \cos v$ where v is the angle between the two vectors, using the Law of Cosines,

$$* \qquad \|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \ \|\mathbf{v}\| \cos v$$



$*$ implies $2\|\vec u\|\|\vec v\|\cos\vartheta = \|\vec u\|^2 + \|\vec v\|^2 - \|\vec u - \vec v\|^2$

$$= \cancel{\vec u \cdot \vec u} + \cancel{\vec v \cdot \vec v} - \cancel{\vec u \cdot \vec u} + 2\vec u \cdot \vec v - \cancel{\vec v \cdot \vec v}$$

$$\Rightarrow \|\vec u\|\|\vec v\| \cos\vartheta = \vec u \cdot \vec v$$

Q.E.D.