5.6-6: Eigenvalues & Dynamical Systems

Math 220: Linear Algebra

A stretch of desert in Northwestern Mexico is populated mainly by two species of animals: coyotes and roadrunners. We wish to model the populations $c(t)$ and $r(t)$ of coyotes and roadrunners *t* years from now if the current populations c_0 and r_0 are known.

From this habitat, the following equations model the transformation of this system from one year to the next, from time *t* to time *t*+1.:

$$\begin{cases} c(t+1) = 0.86c(t) + 0.08r(t) \\ r(t+1) = -0.12c(t) + 1.14r(t) \end{cases}$$

Write this as a matrix product $\vec{x}(t+1) = A\vec{x}(t)$

$$\vec{x}(t+1) = \begin{bmatrix} c(t+1) \\ r(t+1) \end{bmatrix} = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$$

We call $\vec{x}(t)$ the state vector and $\vec{x}(0)$ the initial state vector

This linear transformation is an example of a dynamical system

Suppose we begin with 100 coyotes and 300 road runners, find a close-form formula for $c(t)$ and $r(t)$.

explore: $\begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix} = \underset{\lambda}{1.1} \underset{\text{eigenvec}}{\begin{bmatrix} 100 \\ 300 \end{bmatrix}}$

so $\vec{x}(t) = A^t \vec{x}_0$

$= 1.1^t \begin{bmatrix} 100 \\ 300 \end{bmatrix}$

and $c(t) = 1.1^t \cdot 100$ and $r(t) = 300 \cdot 1.1^t$.

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Suppose we have $c_0 = 200$ and $r_0 = 100$. Find $x(t)$.

explore:

$$\begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix} = 0.9 \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

$\vec{b}_2$

λ eigenvec.

so $\vec{x}(t) = 0.9^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$

Suppose we have $c_0 = r_0 = 1000$. Hint: Write $\vec{x}_0$ in terms of the eigenbasis. Find $\vec{x}(t)$

$$\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = 2 \overset{\vec{b}_1}{\begin{bmatrix} 100 \\ 300 \end{bmatrix}} + 4 \overset{\vec{b}_2}{\begin{bmatrix} 200 \\ 100 \end{bmatrix}}$$

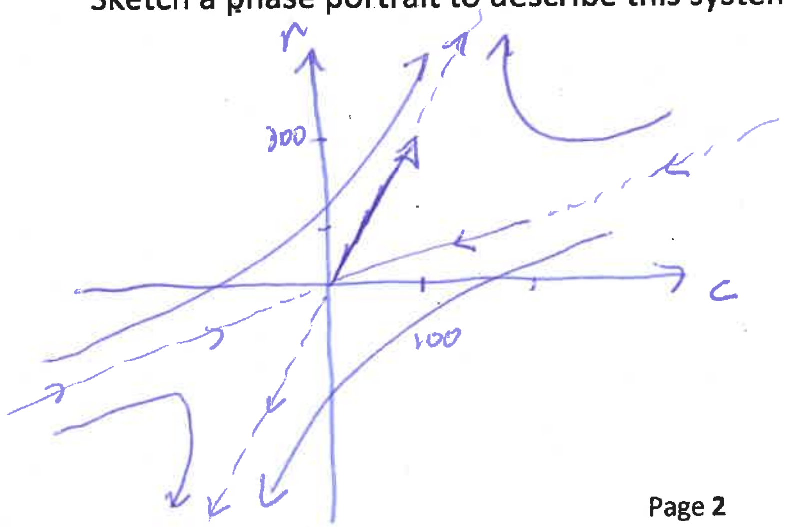
$$\vec{x}(t) = A^t \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$$

$$= A^t (2\vec{b}_1 + 4\vec{b}_2)$$

$$= 2A^t \vec{b}_1 + 4A^t \vec{b}_2$$

$$= 2(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

Sketch a phase portrait to describe this system



note: only the 1st quadrant makes sense in context.

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5.4: Linear Transformations and Dynamical Systems

Here is another example.

Ex 1: Consider $A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$. Since the sum of each column is 1, this linear transformation matrix is called a transition matrix.

a.) Find a closed-form expression for A^t. Hint: Since A is a transition matrix, one of its eigenvalues will be one.

① Find eigenvalues.

solve $0 = \begin{vmatrix} .5-\lambda & .25 \\ .5 & .75-\lambda \end{vmatrix}$

$= (\frac{1}{2}-\lambda)(\frac{3}{4}-\lambda) - \frac{1}{8}$

$= \lambda^2 - \frac{5}{4}\lambda + \frac{1}{4} = \frac{1}{4}(4\lambda^2 - 5\lambda + 1)$

$= \frac{1}{4}(\lambda - 1)(4\lambda - 1)$

$\lambda = 1$ and $\lambda = \frac{1}{4}$

② Find eigenvecs

$\lambda = 1$: $\begin{bmatrix} -.5 & .25 \\ .5 & -.25 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = \frac{1}{4}$: $\begin{bmatrix} .25 & .25 \\ .5 & .5 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

so $A = PDP^{-1}$ w/

$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

And $A^t = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{4})^t \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 + 2(\frac{1}{4})^t & 1 - (\frac{1}{4})^t \\ 2 - 2(\frac{1}{4})^t & 2 + (\frac{1}{4})^t \end{bmatrix}$

b.) If $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $A^t\vec{x}_0$

$A^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^t\left(\frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$= \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{3}(\frac{1}{4})^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

c.) Find the steady-state or equilibrium vector $\vec{x}_{\text{equ}} = \lim_{t\to\infty} A^t\vec{x}_0$

$\lim_{t\to\infty}\left(\frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{3}\underbrace{\cancel{(\tfrac{1}{4})^t}}_{}^{\to 0}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

5.4-6: Eigenvalues and Dynamical Systems

Complex Eigenvalues

Up to this point, we have only discussed real eigenvalues and real-valued vectors (including eigenvectors). But the linear algebra world we have established works over complex numbers of the form $z = a + bi$ where $i^2 = -1$.

Ex 3: Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^n$ of the matrix $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

Then write the eigenvectors $\vec{x}$ in the form $\operatorname{Re}\vec{x} + i\operatorname{Im}\vec{x}$

eigenvalues

solve $0 = \begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$

$= (5-\lambda)(3-\lambda) + 2$

$= \lambda^2 - 8\lambda + 17$

$\Rightarrow \lambda = \dfrac{8 \pm \sqrt{64 - 4(1)(17)}}{2(1)}$

$= 4 \pm i$

eigenvectors.

$A - (4+i)I = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -1-i \\ 1-i & -2 \end{bmatrix}$ $R_2 - (1-i)R_1 \to R_2$

$\sim \begin{bmatrix} 1 & -1-i \\ 0 & 0 \end{bmatrix}$

so the 1st eigenvec is $\vec{v}_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

$A - (4-i)I = \begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}$

and the 2nd is $\vec{v}_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

conclusion: The eigenvecs $\vec{v}_1$ & $\vec{v}_2$ are of the form

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\operatorname{Re}\vec{x}} \pm i \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\operatorname{Im}\vec{x}}$$

Notice that a real-valued matrix can have complex eigenvalues and eigenvectors. Notice further that the eigenvalues and vectors come in conjugate pairs.

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Ex 4: Next we need to unpack the rotation-scaling matrix $C=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

a.) Find the eigenvalues of C.

solve $0 = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix}$

$= (a-\lambda)^2 + b^2$

$= \lambda^2 - 2a\lambda + (a^2+b^2)$

And $\lambda = \dfrac{2a \pm \sqrt{4a^2 - 4(1)(a^2+b^2)}}{2(1)}$

$= \dfrac{2a \pm \sqrt{-4b^2}}{2}$

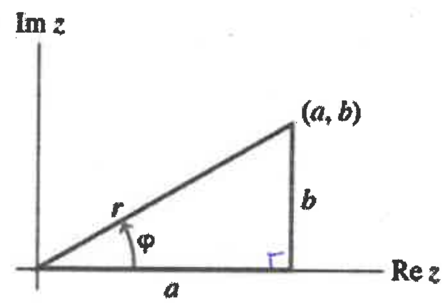
$= a \pm |b|i = a \pm bi$

positive

b.) Let's call $r = |\lambda| = \sqrt{a^2+b^2}$. Then using the picture below, find $\frac{a}{r}$ and $\frac{b}{r}$ in terms of φ.

$\frac{a}{r} = \cos\varphi$

$\frac{b}{r} = \sin\varphi$



Notice These formulas assume a positive b because of $|b|$ in derivation.

So $C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

where $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ is a scaling matrix and $\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ is a rotation matrix.

Ex 5: The matrix $\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$ is a rotation-scaling matrix. Find its eigenvalues, scaling factor, and the angle of rotation φ.

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix} = 5\sqrt{2}\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

so $\lambda = a \pm bi = -5 \pm 5i$

the scaling factor is $|\lambda| = \sqrt{25+25} = 5\sqrt{2}$

$\cos\varphi = -\frac{1}{\sqrt{2}}$ and $\sin\varphi = 1/\sqrt{2}$ so in Quadrant 2 and $\varphi = \frac{3\pi}{4}$

5.4-6: Eigenvalues and Dynamical Systems

This brings us back to the idea of matrix factorization. Recall that if *A* had real eigenvalues and enough linearly independent eigenvectors, then $A = PDP^{-1}$ where the columns of *P* were the eigenvectors and *D* was a diagonal matrix whose diagonal entries were the corresponding eigenvalues.

Similarly, let *A* be a real $2x2$ matrix with a complex eigenvalue $\lambda = a - ib$ $(b \neq 0)$ and an associated eigenvector $\bar{v}$ in $\mathbb{C}^2$. Then $A = PCP^{-1}$ where $P = [\mathrm{Re}\,\bar{v} \quad \mathrm{Im}\,\bar{v}]$ and *C* is the rotation-scaling matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Ex 6: Find an invertible matrix *P* and a matrix *C* of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the matrix $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ has the form $A = PCP^{-1}$

eigenvalues $4 \pm i$, $4 - i$ is of the form $a - bi$
w/ $a = 4$ and $b = 1$.

and the corresponding eigenvec is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

thus $A = PCP^{-1}$ where

$P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$

Trajectories of Dynamical Systems

When we began this lesson, we used a predator-prey example involving coyotes and road runners. We ended that example with a phase portrait that helped us understand the trajectories based upon various initial state vectors.

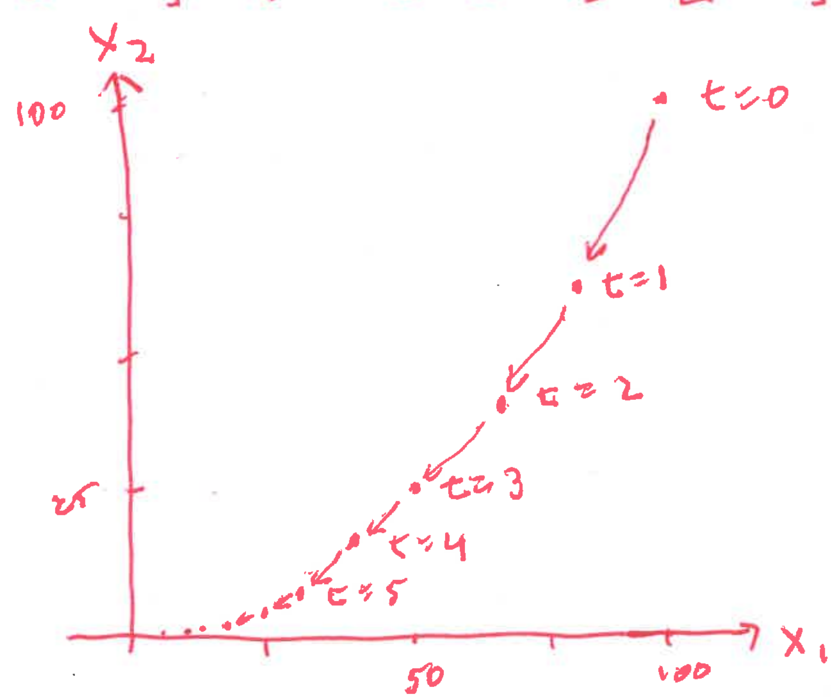
Let's begin by trying to understand how these trajectories work.

Ex 7: Suppose $A=\begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix}$ and $\vec{x}_0=\begin{bmatrix} 100 \\ 100 \end{bmatrix}$, find and plot $\vec{x}(1),\vec{x}(2),\vec{x}(3),\ldots,\vec{x}(10)$

#1* at end of list

t	$t=0$	2	3	4	5	6
$\vec{x}(t)$	$\begin{bmatrix} 100 \\ 100 \end{bmatrix}$	$\begin{bmatrix} 64 \\ 40.96 \end{bmatrix}$	$\begin{bmatrix} 51.2 \\ 26.2 \end{bmatrix}$	$\begin{bmatrix} 40.96 \\ 16.8 \end{bmatrix}$	$\begin{bmatrix} 32.8 \\ 10.7 \end{bmatrix}$	$\begin{bmatrix} 26.2 \\ 6.9 \end{bmatrix}$

t	7	8	9	10	*1*
$\vec{x}(t)$	$\begin{bmatrix} 20.1 \\ 4.4 \end{bmatrix}$	$\begin{bmatrix} 16.8 \\ 2.8 \end{bmatrix}$	$\begin{bmatrix} 13.4 \\ 1.8 \end{bmatrix}$	$\begin{bmatrix} 10.7 \\ 1.2 \end{bmatrix}$	$\begin{bmatrix} 80 \\ 64 \end{bmatrix}$



5.4-6: Eigenvalues and Dynamical Systems

Ex 7: (revisited) $A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix}$, has eigenvalues $\lambda_1 = 0.8$ and $\lambda_2 = 0.64$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So if $\vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2$, then $\vec{x}_k = c_1(0.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

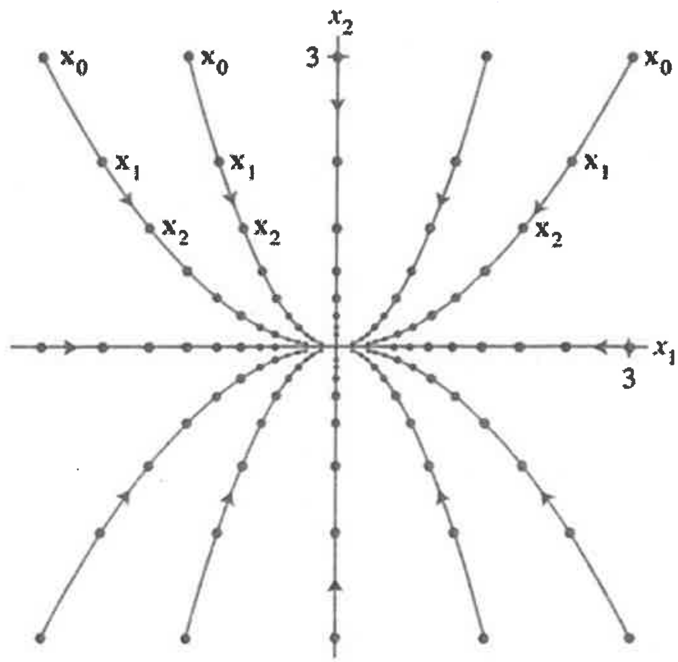


FIGURE 1 The origin as an attractor.

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Ex 8: Suppose $A=\begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$. What are the eigenvalues and eigenvectors?

eigenvalues 1.44 XXXXXXXXXX and 1.2

eigen vecs $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

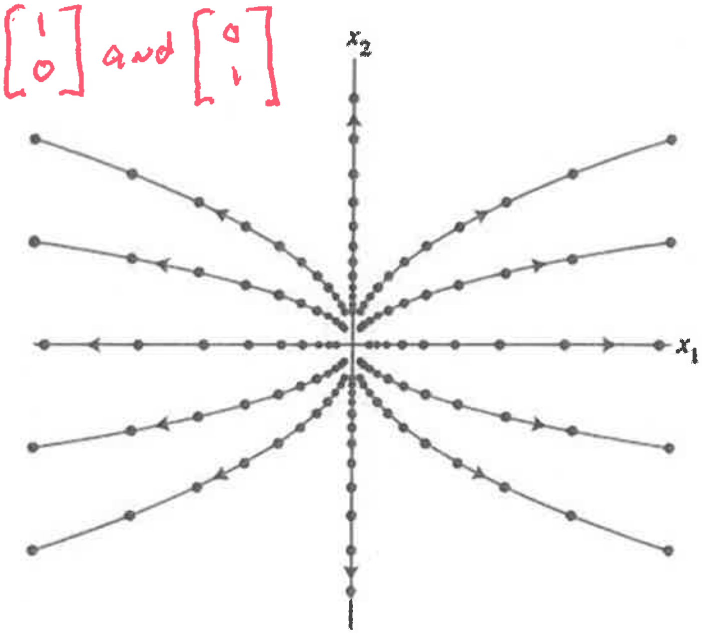
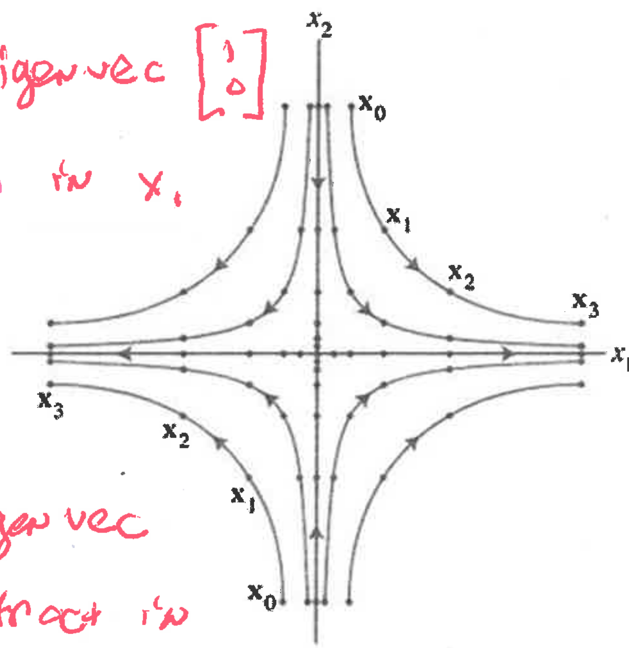


FIGURE 2 The origin as a repeller.

Ex 9: Suppose $A=\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$. Here is a phase portrait for it.

$\lambda = 2$ w/ eigenvec $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so repel in x_1 direction.



$\lambda = 0.5$ w/ eigenvec $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so attract in x_2 direction.

FIGURE 3 The origin as a saddle point.

5.4-6: Eigenvalues and Dynamical Systems

Question: In the previous examples, we have focused on diagonal matrices? Is this reasonable? Is it overly simplistic? Explain.

If there are enough eigenvecs, then a dynamical system is diagonalizable. we can think of P & P^{-1} as giving a change of basis to one where the diagonal Matrix represents the transformation

Ex 10: Show that the origin is a saddle point for the solutions of $\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$.

solve $0 = (\frac{5}{4} - \lambda)(\frac{5}{4} - \lambda) - \frac{9}{16}$

$= \lambda^2 - \frac{5}{2}\lambda + 1$

$\Rightarrow 0 = 2\lambda^2 - 5\lambda + 2$

$\lambda = \frac{5 \pm \sqrt{25 - 4(2)(2)}}{2(2)}$

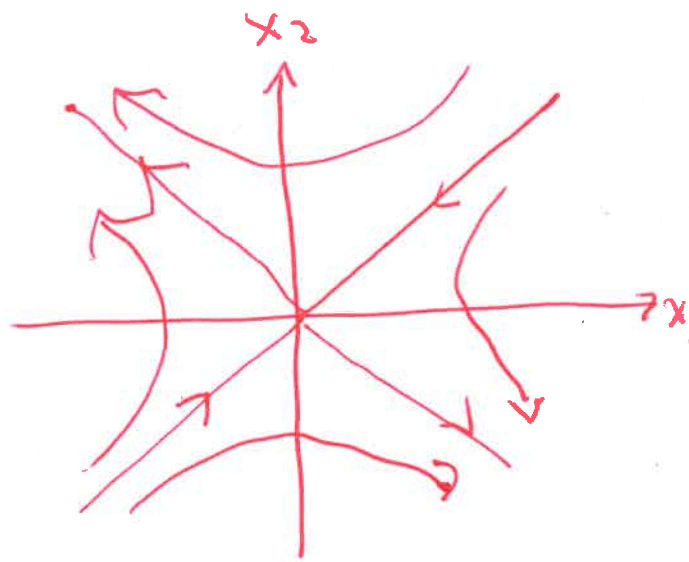
$= 2$ OR $\frac{1}{2}$

$A - 2I = \begin{bmatrix} -.75 & -.75 \\ -.75 & -.75 \end{bmatrix}$

so the eigenvec is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A - \frac{1}{2}I = \begin{bmatrix} .75 & -.75 \\ -.75 & .75 \end{bmatrix}$

so the eigenvec is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



so it is

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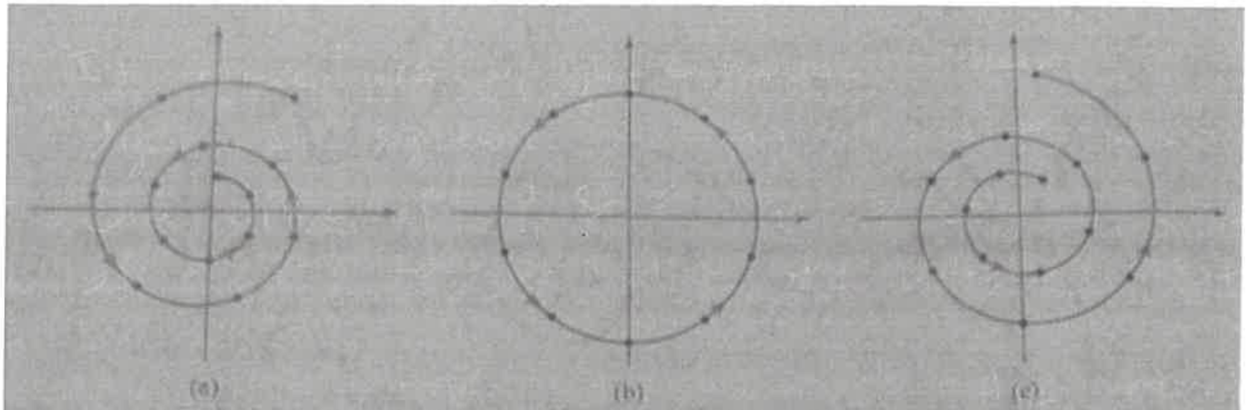
Phase portraits get more interesting with complex eigenvalues



(a) $r < 1$: trajectories spiral inward. (b) $r = 1$: trajectories are circles. (c) $r > 1$: trajectories spiral outward.

Ex 11: Consider the dynamical system and sketch the trajectory of $\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

solve $0 = (3-\lambda)(-1-\lambda) + 5$

$= \lambda^2 - 2\lambda + 2$

$\Rightarrow \lambda = \dfrac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$

$= \dfrac{2 \pm 2i}{2}$

$= 1 \pm i$ (Note: $|\lambda| = \sqrt{2}$)

$A - (1-i)I = \begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -2+i \\ 2+i & -5 \end{bmatrix}$

so the eigenvec is $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

or $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

so $A = \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{P^{-1}}$

$C = \sqrt{2} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$

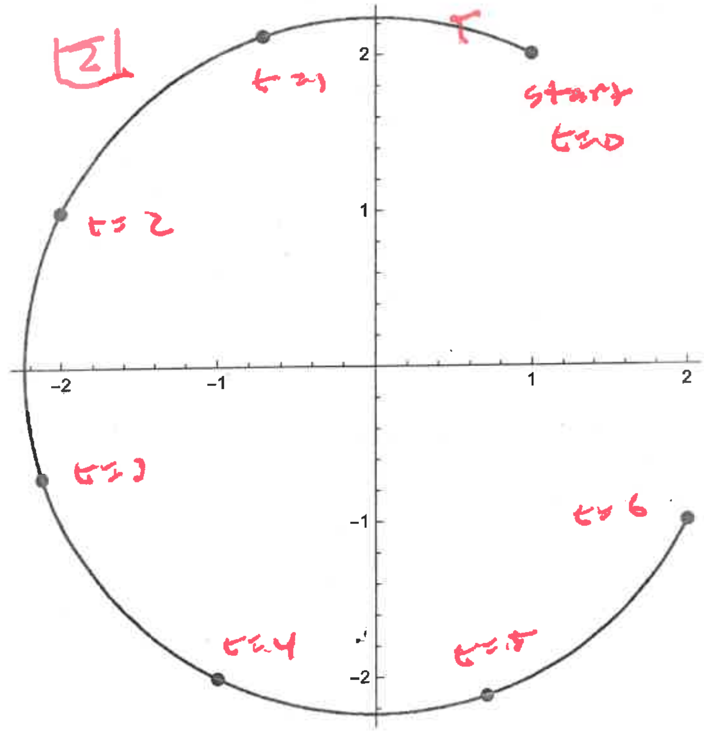
Now to sketch the trajectory.

① Find $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in terms of the $\mathrm{Re}\,\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathrm{Im}\,\vec{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ basis. $P^{-1}\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

② What does the rotation do?

$$\begin{bmatrix} \cos\frac{\pi}{4}t & -\sin\frac{\pi}{4}t \\ \sin\frac{\pi}{4}t & \cos\frac{\pi}{4}t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

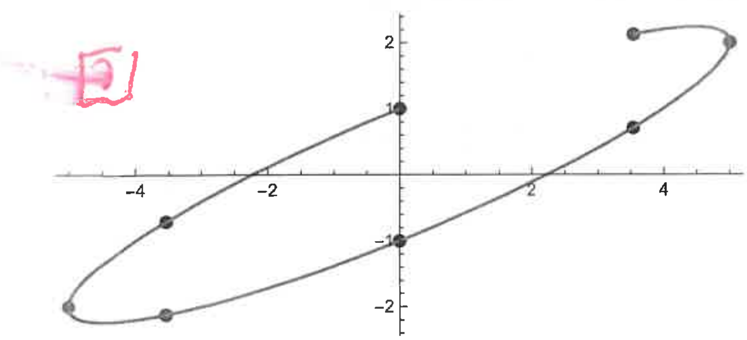
It starts @ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and rotates $45°$ ccw w/ each step.



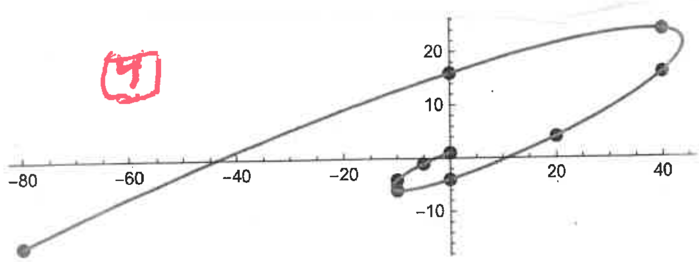
③ Then we go back to the original coordinates

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{4}t & -\sin\frac{\pi}{4}t \\ \sin\frac{\pi}{4}t & \cos\frac{\pi}{4}t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This puts us on an elliptical path.



④ Finally, we add the scaling factor which causes the trajectory to spiral out.



$$\vec{x}(t) = (\sqrt{2})^{t}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \cdot$$

$$\hookrightarrow \begin{bmatrix} \cos\frac{\pi}{4}t & -\sin\frac{\pi}{4}t \\ \sin\frac{\pi}{4}t & \cos\frac{\pi}{4}t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$