4.1: Vector Spaces and Subspaces

Math 220: Linear Algebra

Definition
A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors **u**, **v**, and **w** in V and for all scalars c and d.

1. The sum of **u** and **v**, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector **0** in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each **u** in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of **u** by c, denoted by $c\,\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

It also follows that

$$0\mathbf{u}=\mathbf{0} \qquad (1)$$

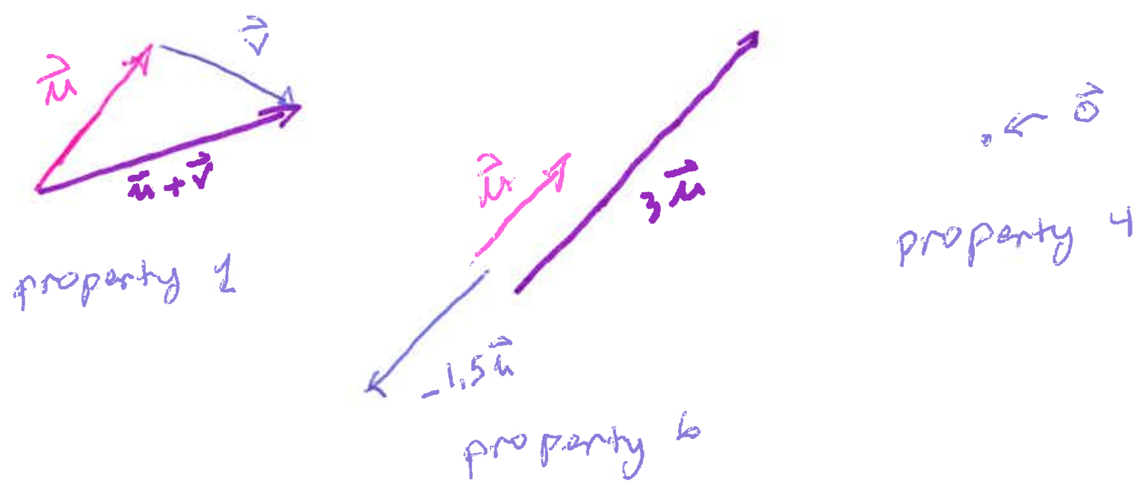
$$c\mathbf{0}=\mathbf{0} \qquad (2)$$

$$-\mathbf{u}=(-1)\mathbf{u} \qquad (3)$$

The spaces $\mathbb{R}^n$ for $n \geq 1$ are the best examples of vector spaces. We will picture $\mathbb{R}^3$ ($\mathbb{R}^2$) for much of our discussion of vector spaces.

Ex 1:

Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule (from Section 1.3), and for each **v** in V, define c **v** to be the arrow whose length is $|c|$ times the length of **v**, pointing in the same direction as **v** if $c \geq 0$ and otherwise pointing in the opposite direction. (See Figure 1.) Show that V is a vector space. This space is a common model in physical problems for various forces.



Read Example 3 on page 193

Ex 2: Discuss whether the set P_n of polynomials of degree at most n is a vector space.

we need polynomials in $\mathbb{P}_n$ and scalars w/ which to work.

$p_n = a_0 + a_1 x + \dots + a_n x^n$

$s_n = b_0 + b_1 x + \dots + b_n x^n$

and scalar $c \in \mathbb{R}$

① $p_n + s_n = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n \in \mathbb{P}_n$

properties 2, 3, 7, 8, 9, 10 come from properties of real numbers

④ $0 + 0x + \dots + 0x^n = 0 \in \mathbb{P}_n$

⑤ If $a_0 = -b_0, \dots, a_n = -b_n \Rightarrow s_n = -p_n \in \mathbb{P}_n$

⑥ $cp_n = c(a_0 + a_1 x + \dots + a_n x^n) = (ca_0) + (ca_1)x + \dots (ca_n)x^n \in \mathbb{P}_n$

$\therefore$ $\mathbb{P}_n$ is a vector space.

Read Example 5 on page 194

Definition

A **subspace** of a vector space *V* is a subset *H* of *V* that has three properties:

a. The zero vector of *V* is in *H*.

b. *H* is closed under vector addition. That is, for each **u** and **v** in *H*, the sum $\mathbf{u} + \mathbf{v}$ is in *H*.

c. *H* is closed under multiplication by scalars. That is, for each **u** in *H* and each scalar *c*, the vector *c* **u** is in *H*.

Note ; **every subspace is itself a Vector space.**

The set of just the zero vector in a vector space *V* is a subspace of V called the zero subspace and written $\{\vec{0}\}$.

Ex 3: Let P be the set of all polynomials and a subspace of the set of all real-valued functions. Is P_n is a subspace of *P*.

Let $f_1 = a_0 + a_1 x + \dots + a_n x^n$ and $f_2 = b_0 + b_1 x + \dots + b_n x^n$ and c be a scalar. Note: $f_1, f_2 \in P_n$

(a) $0 \in P_n$ since $0 = 0 + 0x + \dots + 0x^n$.

(b) $f_1 + f_2 = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n \in P_n$

(c) $cf_1 = ca_0 + ca_1 x + \dots + ca_n x^n \in P_n$

$\therefore$ P_n is a subspace of P.

Notation: "$\subset$" subset

"$\not\subset$" Not a subset

Ex 4: The vector space $\mathbb{R}^2$ is NOT a subspace of $\mathbb{R}^3$, but *H* is. Discuss.

$$\begin{bmatrix} x \\ y \end{bmatrix} \notin \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}$$

so $\mathbb{R}^2 \not\subset \mathbb{R}^3$

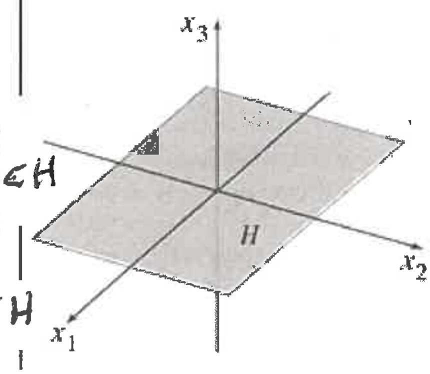
$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

Is H a subspace?

(a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$ when $s = t = 0$

(b) $\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ 0 \end{bmatrix} \in H$

(c) $\vec{u} \in H$ and $c \in \mathbb{R} \Rightarrow c\vec{u} = c\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ 0 \end{bmatrix} \in H$



$\therefore$ H is a subspace.

What about a plane not through the origin? Or a line in $\mathbb{R}^2$ not through the origin? Are they Subspaces? (of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively).

No, they do not include $\vec{0}$.

Ex 5: Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space *V*, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that *H* is a subspace of *V*.

(a) $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$

(b) $\vec{u}, \vec{w} \in \text{Span}\{v_1, v_2\} \Rightarrow \exists\, a, b, c, d$ s.t. $\vec{u} = a\vec{v}_1 + b\vec{v}_2$ and $\vec{w} = c\vec{v}_1 + d\vec{v}_2$

$\Rightarrow \vec{u} + \vec{w} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_1 + d\vec{v}_2$

$= (a+c)\vec{v}_1 + (b+d)\vec{v}_2$

$\in \text{Span}\{\vec{v}_1, \vec{v}_2\}$

(c) $\vec{u} \in \text{Span}\{v_1, v_2\}$ and $c \in \mathbb{R} \Rightarrow \exists\, a, b$ s.t. $\vec{u} = a\vec{v}_1 + b\vec{v}_2$

$\Rightarrow c\vec{u} = (ca)v_1 + (cb)v_2$

$\in \text{Span}\{\vec{v}_1, \vec{v}_2\}$

$\therefore$ H is a subspace.

Theorem 1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Thm 1 is an extension of the previous example.

We call this subspace the subspace spanned by $\{v_1, \ldots v_p\}$

And for any subspace H, we call the set $\{v_1, \ldots v_p\}$ such that $H = \text{Span}\{v_1, \ldots v_p\}$,

the spanning set.

Ex 6: Let H be the set of all vectors of the form $\begin{bmatrix} a \\ 3a+b \\ b \\ a-2b \end{bmatrix}$ where a and b are arbitrary scalars. Show that H is a subspace of $\mathbb{R}^4$

$$\begin{bmatrix} a \\ 3a+b \\ b \\ a-2b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}}_{\vec{v}_2}$$

Thus $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$

$\therefore$ by Thm 1 H is a subspace of $\mathbb{R}^4$.

We can think of the vectors in a spanning set as the "handles" that define a subspace H, and allow us to hold it and work with it.

Ex 7: For what value(s) of h will $\mathbf{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

(This is the same example in the text from 1.3 – now with the context of subspaces.)

solve $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{y}$

$$\Rightarrow \begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \begin{matrix} R_2 + R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} R_3 - 3R_2 \to R_3$$

$$\sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

$\vec{y}$ is in the subspace when the system is consistent. That is when $h = 5$ and $\vec{y} = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$

Practice Problems

1. Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector $\mathbf{u}$ in H and a scalar c such that $c\,\mathbf{u}$ is not in H.)

If $s = 1$, the vector is $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Let's try $2\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ $\begin{matrix} \leftarrow s = 2 \\ \leftarrow s = \frac{12}{5} \end{matrix}$ } not the same.

3. An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times3}$, the vector space of 3×3 matrices.

(a) $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0^T \in S$

(b) $A + B = A^T + B^T = (A+B)^T \in S$

(c) $cA = cA^T = (cA)^T \in S$

$\therefore$ S is a subspace