3.1 & 3.2: Determinants

Math 220: Linear Algebra

Although out of fashion, determinants played a large role in the early development of linear algebra. Four uses of determinants include the following: Determinants help us "determine" if a system of linear equations has a unique solution. They are a mechanism to "determine" whether the inverse of a matrix exists (this would have come later). They may be geometrically interpreted as the scaling factor of a linear transformation. And the determinant is also a calculating mechanism used elsewhere in math to find things such as the cross-product (Calculus III), Jacobian (Calculus IV), and the Wronskian (Differential Equations).

As to why they have fallen out of favor? Well they are computationally expensive even with modern technology. So we have adopted other ways to accomplish their original purpose.

Their primary reason for being in this course is that they are needed for our development of the eigenvalue and eigenvector in a subsequent chapter.

Ex 1: If $A=\begin{bmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ find $\det A$ which is also notated $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix}$

$$\det A = 0\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix}$$

$$= 0 - 4(5) + 1(21)$$

$$= 1$$

Ex 2: Calculate $\begin{vmatrix} 0 & 1 & 4 \\ 5 & 0 & -3 \\ 2 & 1 & 3 \end{vmatrix}$ by expanding across the second column.

$$= -1\begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ 5 & -3 \end{vmatrix}$$

$$= -21 + 0 + 20$$

$$= -1$$

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Ex 3: Compute the determinant: $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -4 \end{vmatrix}$

$$= 1\begin{vmatrix} -2 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -4 \end{vmatrix} = 1(-2)\begin{vmatrix} 3 & 0 \\ 4 & -4 \end{vmatrix} = 1(-2)(3)(-4)$$

Theorem 2

If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Ex 4: Compute the determinant: $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$

$$= 2\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} = 2(+5)\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 2(5)(2) = 20$$

Theorem 3: Row Operations

Let A be a square matrix

a. If a multiple of one row of A (old) is added to another row to produce a matrix B (new), then $\det A = \det B$.

b. If two rows of A (old) are interchanged to produce B (new), then $\det A = -\det B$.

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c. If one row of A (old) is multiplied by k to produce B (new), then $\det A = \frac{1}{k} \det B$

Ex 5: Find the determinant by first row-reducing to echelon form.

$$\begin{vmatrix} 3 & 3 & -3 \\ 2 & -3 & -5 \\ 3 & 4 & -4 \end{vmatrix} \quad \tfrac{1}{3}R_1 \to R_1 \quad = 3 \begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & -5 \\ 3 & 4 & -4 \end{vmatrix} \quad \begin{matrix} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{matrix} \quad = 3 \begin{vmatrix} 1 & 1 & -1 \\ 0 & -5 & -3 \\ 0 & 1 & 5 \end{vmatrix}$$

$R_2 \leftrightarrow R_3$

$$= -3 \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 5 \\ 0 & -5 & -3 \end{vmatrix} \quad R_3 + 5R_2 \to R_3 \quad = -3 \begin{vmatrix} 1 & 1 & - \\ 0 & 1 & 5 \\ 0 & 0 & -8 \end{vmatrix} = 24$$

Ex 6: Find the determinant by first row-reducing to echelon form.

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} \quad \begin{matrix} R_2 + 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{matrix} \quad = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} \quad \begin{matrix} R_3 + 4R_2 \to R_3 \\ R_4 + 4R_2 \to R_4 \end{matrix}$$

$$= \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix} \quad R_4 - R_3 \to R_4 \quad = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Let's consider two different triangular matrices and their invertibility. The focus on triangular matrices is reasonable as we learned in a previous section that row operations do not impact the invertibility of matrices.

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \qquad U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U \neq 0$ $\qquad\qquad$ $\det U = 0$

u is invertible $\qquad\qquad$ u is <u>not</u> invertible

Theorem 4
A square matrix A is invertible if and only if $\det A \neq 0$.

Ex 7: Revisiting (Ex 6:), at what point could we have stopped?

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = 0$$

same

Theorem 5

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Theorem 6 Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Ex 8: Verify Thm 6 for $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 3 & 6 \\ -1 & -2 \end{vmatrix} = -6-(-6) = 0$$

$$\det(B) = \begin{vmatrix} 4 & 3 \\ -1 & -3 \end{vmatrix} = -12-(-3) = -9$$

$$AB = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -2 & 3 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 6 & -9 \\ -2 & 3 \end{vmatrix} = 18-(+18) = 0$$

Thus $\underbrace{\det AB}_{0} = \underbrace{(\det A)}_{0}\underbrace{(\det B)}_{-9}$

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Practice Problems

1. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.

$R_2 - 2R_1 \to R_2$

$R_4 + 3R_1 \to R_4$

$$= \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix}$$

same row.

$$= 0$$

2. Use a determinant to decide if $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

Let $A = [v_1\ v_2\ v_3]$ and find $\det(A)$.

$$\begin{vmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix} = 5\begin{vmatrix} 3 & -7 \\ -5 & 5 \end{vmatrix} + 3\begin{vmatrix} -7 & -7 \\ 9 & 5 \end{vmatrix} + 2\begin{vmatrix} -7 & 3 \\ 9 & -5 \end{vmatrix}$$

$$= 5(-20) + 3(28) + 2(8)$$

$$= 0$$

so A is not invertible and its cols are $\therefore$ not linearly independent

3. Let A be an $n \times n$ matrix such that $A^2 = I$. Show that $\det A = \pm 1$.

Let $A^2 = I$

$\Rightarrow \det(A^2) = \det(I)$

$\Rightarrow \det(A)\det(A) = 1$

$\Rightarrow [\det(A)]^2 = 1$

$\Rightarrow \det(A) = \pm 1$