1.7: Linear Independence

Math 220: Linear Algebra

Definition

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

$\to x_1 = x_2 = \ldots = x_p = 0$

has only the $\boxed{\text{trivial solution}}$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Ex 1: Determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. If not, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \text{ (free)} \end{aligned} \quad \Rightarrow \quad \vec{x} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

so, for example, if $x_3 = 1$ this tells us $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$. This is a non-trivial solution to the homogeneous equation so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent.

1.7: Linear Independence

This is a big deal!

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Ex 2: Determine whether the columns of the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -3 & 1 & -2 \end{bmatrix}$ are linearly independent.

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ -3 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}\right\}$$ the trivial solution is the only solution so the columns of A are linearly independent.

Ex 3: What about? **Linearly Dependent or Independent? Why?**

(L.D.) (L.I.)

$\{\mathbf{v}\}$, not the zero vector — L.I. since $c\vec{v} = \vec{0} \Rightarrow c = 0$

$\{\mathbf{0}\}$ — L.D. since $c \cdot \vec{0} = \vec{0}$ for $c \in \mathbb{R}$

$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$ — L.D. $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

that is $3\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$ — L.I. $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

only the trivial solution.

1.7: Linear Independence

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Theorem 7 Characterization of Linearly Dependent Sets

claim: An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Proof:

($\Rightarrow$) Assume S is linearly dependent.

If $\vec{v}_1 = \vec{0}$ then it is a linear combinaton of the other vectors: $0\vec{v}_2 + \dots + 0\vec{v}_p = \vec{v}_1$.

If $\vec{v}_1 \neq \vec{0}$ then $c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1} + c_j\vec{v}_j + c_{j+1}\vec{v}_{j+1} + \dots + c_p\vec{v}_p = \vec{0}$ where not all of $c_1, \dots, c_p$ are zero. Suppose $c_j \neq 0$ and $c_{j+1} = \dots = c_p = 0$.

$\Rightarrow c_1\vec{v}_1 + \dots + c_j\vec{v}_j + 0\vec{v}_{j+1} + \dots + 0\vec{v}_p = \vec{0}$

$\Rightarrow \vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \dots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1}$

so a vector is a linear combination of the others.

($\Leftarrow$) Suppose one vector, call it $\vec{v}_j$, is a linear combination of the others. (Note: we can always reorder if need be to make this true).

$\Rightarrow \exists\, c_1, \dots, c_{j-1} \in \mathbb{R}$ s.t. $\vec{v}_j = c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1}$

$\Rightarrow c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j + 0\vec{v}_{j+1} + \dots + 0\vec{v}_p = \vec{0}$

not zero

$\Rightarrow$ there is a non-trivial solution to the homogeneous equation and S is linearly dependent.

$\therefore$ S is L.D. iff one or more vectors in S is a linear combination of the others

1.7: Linear Independence

claim:

Ex 4: Given the set of vectors $\{\mathbf{u},\mathbf{v},\mathbf{w}\} \in \mathbb{R}^3$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent, explain why vector $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ if and only if $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is linearly dependent.

proof.

suppose $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ w/ $\vec{u}, \vec{v}$ L.I. and $w \in \text{Span}\{u, v\}$

$\Leftrightarrow \quad w = a\vec{u} + b\vec{v}$

$\Leftrightarrow \quad \vec{0} = a\vec{u} + b\vec{v} - 1\vec{w}$ which is a non-trivial solution to the homogeneous equation and thus $\{u, v, w\}$ is linearly dependent.

Q.E.D.

Theorem 8
If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof:

Let $\vec{v}_1, \ldots, \vec{v}_p \in \mathbb{R}^n$ w/ $p > n$ be given.

The main point is that too many vectors guarantees linear dependence.

less rows

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & & | \end{bmatrix}_{n \times p} \sim \begin{bmatrix} \text{at most} \\ n \text{ pivots} \\ \\ \end{bmatrix}$$

more ~~XXX~~ columns

$\Rightarrow$ free variables

$\Rightarrow A\vec{x} = \vec{0}$ has non-trivial solutions

$\therefore \vec{v}_1, \ldots, \vec{v}_p$ are L.D.

Ex 5: Using Theorem 8, create a set of vectors in $\mathbb{R}^3$ that is linearly dependent, and don't automatically make some of the vectors obvious multiples or combinations of the others.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Any 4+ vectors in $\mathbb{R}^3$ do the trick.

1.7: Linear Independence

Theorem 9
If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof:
Let $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^N$ contain $\vec{0}$.
WLOG assume $\vec{v}_1 = \vec{0}$
$\Rightarrow 1\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_p = \vec{0}$
$\Rightarrow$ there is a non-trivial solution to the homogeneous equation.
$\therefore$ the vectors are L.D.

Ex 6: Determine by inspection if the give set is linearly dependent.

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ L.D. (too many vecs)

b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$ L.D. (includes $\vec{0}$)

c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$ L.I. (Not scalar multiples)

1.7: Linear Independence

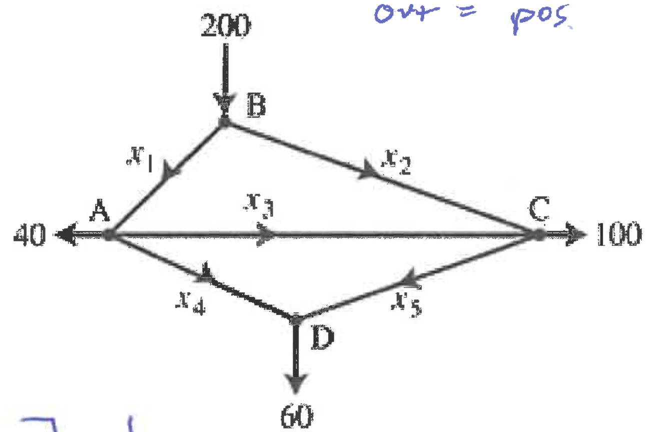
Ex 7: Network flow exercise from 1.6 (we did a chemistry example previously).

a) Find the general traffic pattern in the freeway network shown in the figure. (Flow rates are in cars/minute)

b) Describe the general traffic pattern when the road whose flow is x_4 is closed.

c) When $x_4 = 0$, what is the minimum value of x_1?

IN = neg
OUT = pos



A: $-x_1 + x_3 + x_4 + 40 = 0$

B: $x_1 + x_2 - 200 = 0$

C: $-x_2 - x_3 + x_5 + 100 = 0$

D: $-x_4 - x_5 + 60 = 0$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & -40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & -1 & 0 & 1 & -100 \\ 0 & 0 & 0 & -1 & -1 & -60 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & 1 & 100 \\ 0 & \boxed{1} & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & \boxed{1} & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)

$$\Rightarrow \begin{aligned} x_1 &= 100 + x_3 - x_5 \\ x_2 &= 100 - x_3 + x_5 \\ x_3 &= x_3 \text{ (free)} \\ x_4 &= 60 - x_5 \\ x_5 &= x_5 \text{ (free)} \end{aligned}$$

(b) If $x_4 = 0$

$x_1 = 40 + x_3$

$x_2 = 160 - x_3$

$x_3 = x_3$ (free)

$x_4 = 0$

$x_5 = 60$

(c) since $x_3 \geq 0$ the minimum of $x_1 = 40$.