1.3: Vector Equations

Math 220: Linear Algebra

Vectors in $\mathbb{R}^2$

A matrix with one column is called a <u>column</u> <u>vector</u> or <u>vector</u>

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

Vectors are <u>equal</u> if and only if the corresponding entries are equal.

The sum of the vectors $\mathbf{u}$ and $\mathbf{v}$ is the vector $\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 + v_1 \\ 3 + v_2 \end{bmatrix}$

The scalar multiple of vector $\mathbf{w}$ by a real number c is the vector $c\mathbf{w}$ where each <u>component</u> of $\mathbf{w}$ is multiplied by c. $\quad c\vec{w} = c \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} cw_1 \\ \vdots \\ cw_N \end{bmatrix}$

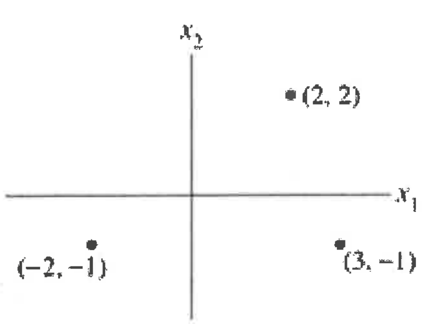
Ex 1: Given $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ find

a) $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

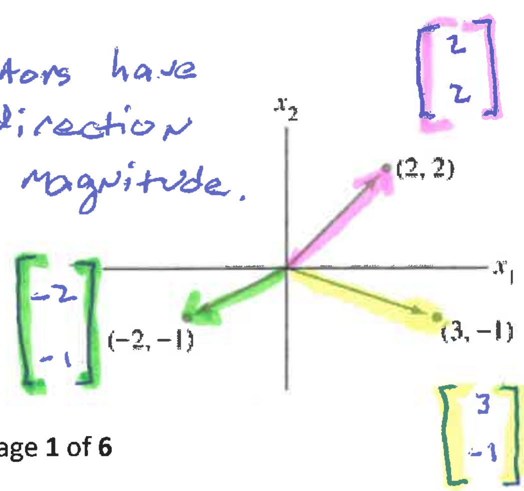
b) $3\mathbf{u} = 3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$

c) $2\mathbf{u}-5\mathbf{v} = 2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} - 5 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ -24 \end{bmatrix}$

Geometric Descriptions of $\mathbb{R}^2$



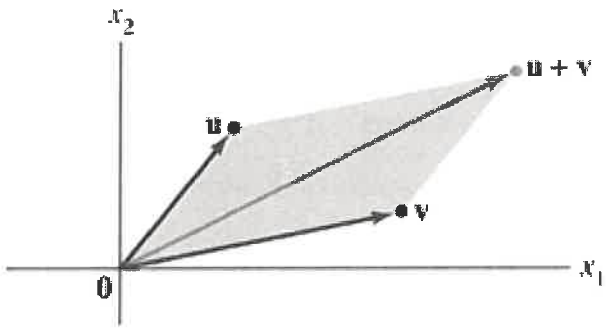
vectors have a direction and magnitude.



1.3: Vector Equations

Parallelogram Rule for Addition

If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See Figure 3.



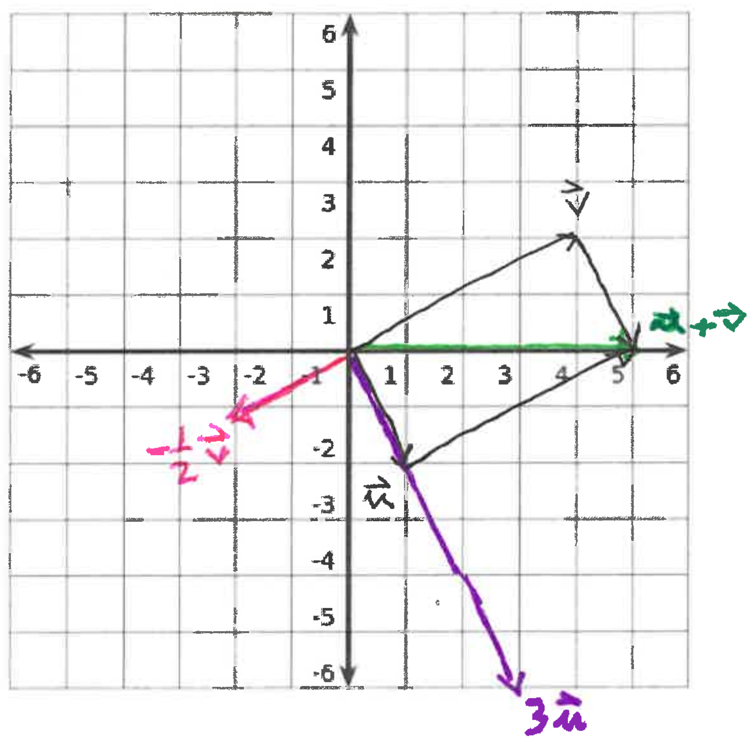
Ex 2: Given $\mathbf{u}=\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\2\end{bmatrix}$,

draw their vectors and the following.

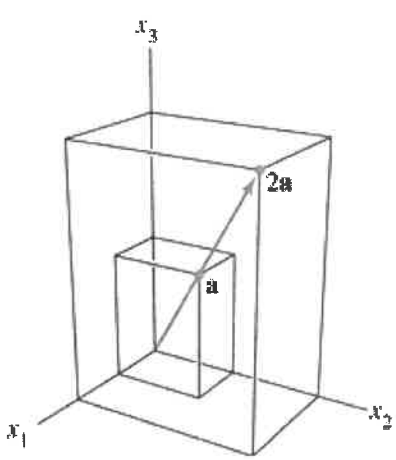
a) $\mathbf{u}+\mathbf{v} = \begin{bmatrix}5\\0\end{bmatrix}$

b) $3\mathbf{u} = \begin{bmatrix}3\\-6\end{bmatrix}$

c) $-\dfrac{1}{2}\mathbf{v} = \begin{bmatrix}-2\\-1\end{bmatrix}$



Vectors in $\mathbb{R}^3$



Vectors in $\mathbb{R}^n$

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}$$

The Zero vector has entries of all zero,

denoted by $\mathbf{0}$ or

$$\vec{0} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

Prove (i) and (v)

(i) claim: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

proof.

Let $\vec{u},\vec{v}\in\mathbb{R}^n$ be given.

$\Rightarrow$ there exist $u_1,\dots,u_n$ and $v_1,\dots,v_n$ s.t.

$$\vec{u}+\vec{v}=\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}+\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}$$

$$=\begin{bmatrix}u_1+v_1\\ \vdots\\ u_n+v_n\end{bmatrix}$$

commutative property of real numbers

$$=\begin{bmatrix}v_1+u_1\\ \vdots\\ v_n+u_n\end{bmatrix}$$

$$=\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}+\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}$$

$=\vec{v}+\vec{u}$

$\therefore\ \vec{u}+\vec{v}=\vec{v}+\vec{u}$

(v.) -----> XXX claim: $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$

proof.

Let $u, v\in\mathbb{R}^n$ and scalar c be given.

$\Rightarrow$ there exists $u_1,\dots,u_n$ and $v_1,\dots,v_n$ s.t.

$$c(\vec{u}+\vec{v})=c\left(\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}+\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}\right)$$

$$=c\begin{bmatrix}u_1+v_1\\ \vdots\\ u_n+v_n\end{bmatrix}$$

$$=\begin{bmatrix}c(u_1+v_1)\\ \vdots\\ c(u_n+v_n)\end{bmatrix}$$

distributive property of addition of real numbers,

$$=\begin{bmatrix}cu_1+cv_1\\ \vdots\\ cu_n+cv_n\end{bmatrix}$$

$$=\begin{bmatrix}cu_1\\ \vdots\\ cu_n\end{bmatrix}+\begin{bmatrix}cv_1\\ \vdots\\ cv_n\end{bmatrix}$$

$$\Leftrightarrow \quad = c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + c\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= c\vec{u} + c\vec{v}$$

$$\therefore \quad c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}.$$

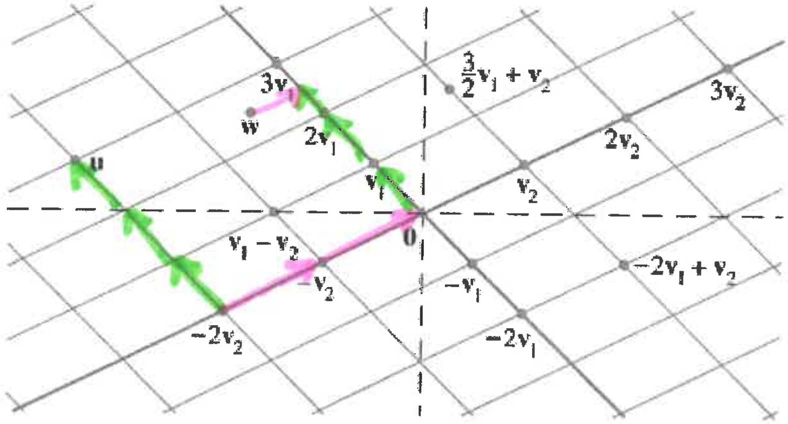
Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \dots, c_p$, the vector **y** defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with **weights** $c_1, \dots, c_p$.

Ex 3: The figure identifies selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



$\vec{u} = 3\vec{v}_1 + (-2)\vec{v}_2$

$\vec{w} = \frac{5}{2}\vec{v}_1 + \left(-\frac{1}{2}\right)\vec{v}_2$

Ex 4: Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

solve $c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c_1 + 5c_3 \\ -2c_1 + c_2 - 6c_3 \\ 2c_2 + 8c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + 5c_3 &= 2 \\ -2c_1 + c_2 - 6c_3 &= -1 \\ 2c_2 + 8c_3 &= 6 \end{aligned}$$

write as an augmented matrix

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

RREF on the calculator

$$\sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_1 + 5c_3 = 2$

$c_2 + 4c_3 = 3$

$$\Rightarrow \begin{array}{l} c_1 = 2 - 5c_3 \\ c_2 = 3 - 4c_3 \\ c_3 = c_3 \text{ (free)} \end{array} \quad \text{OR} \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$$

$\therefore$ $\vec{b}$ can be written as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. There are an infinite ways

One example when $c_3 = 2$ (my choice)

is $-8\vec{a}_1 + -5\vec{a}_2 + 2\vec{a}_3 = \vec{b}$

1.3: Vector Equations

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}] \qquad (5)$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

Definition

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.

this means there exist $c_1, \dots, c_p$ s.t. $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{b}$

$$\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}?$$

translates $\vec{b}$ is contained in the span or $\vec{b}$ is an element of the span

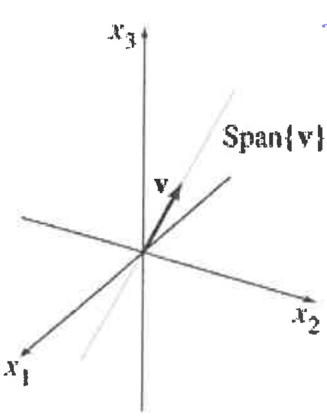
Every scalar multiple of individual vectors, $c\mathbf{v}_k$ $\in$ span because

$$c\vec{v}_k = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_{k-1} + c\vec{v}_k + 0\vec{v}_{k+1} + \dots + 0\vec{v}_p$$

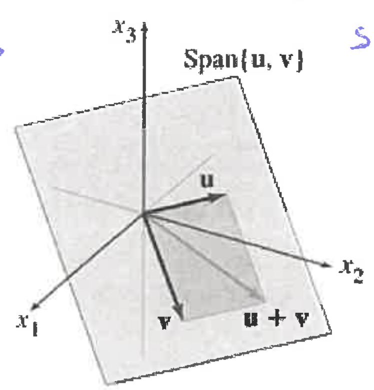
for $k = 1, 2, \dots, p$.

Geometric Description of Span $\{\mathbf{v}\}$ and Span $\{\mathbf{u}, \mathbf{v}\}$

provided $\vec{u}, \vec{v}$ are not parallel, their span is a "plane."



the span of one non-zero vector is a "line."



Ex 5: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 10 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$. $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane in $\mathbb{R}^3$.

Is **b** in that plane?

$$\text{rref}\left(\begin{bmatrix} 1 & 3 & -1 \\ 3 & 10 & 4 \\ -2 & -4 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{inconsistent system.}$$

$\therefore$ No, $\vec{b} \notin \text{Span}\{\vec{a}_1, \vec{a}_2\}$

Ex 6: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$

For what value(s) of h is **y** in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$?

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \quad R_3 + 2R_1 \to R_3$$

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \quad R_3 - 2R_2 \to R_3$$

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{bmatrix}$$

in order to be consistent $7 + 2h = 0$

$\Rightarrow h = -7/2$

so $\vec{y}$ is in the plane iff $h = -7/2$