Assessment 5
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Math 220

Name: ______________________________

No work = no credit

...there is no study in the world which brings into more harmonious action all the faculties of the mind than [mathematics], ... or, like this, seems to raise them, by successive steps of initiation, to higher and higher states of conscious intellectual being....

James Joseph Sylvester
1814 - 1897 (English mathematician)

Warm-ups (1 pt each):
Note: Assume $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^2$

$\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}$ $\qquad$ $\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$ $\qquad$ $\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}$

$\vec{e}_1\vec{e}_1^{\,T} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ $\qquad$ $\vec{e}_1^{\,T}\vec{e}_1 = \begin{bmatrix}1\end{bmatrix}$ $\qquad$ $\vec{e}_2\vec{e}_1^{\,T} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$

1.) (1 pt) The quote above is by Sylvester who was one of the inventers of linear algebra. Sylvester seems to think that math was the key to enlightenment. Do you agree? Why or why not. Answer using complete English sentences.

Math helps, but is not the key. It is hubris to think that one's field is the be-all, end-all.

2.) (5 pts) Consider $A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a.) Find a basis for $\operatorname{Col} A$.

$\left\{ \begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\-3\\-8\\7\end{bmatrix} \right\}$

b.) Find $\operatorname{Nul} A$ =

$\operatorname{span}\left\{ \begin{bmatrix}3\\4\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\5\\0\\2\\1\end{bmatrix} \right\}$

3.) (5 pts) Prove the Unique Representation Theorem which states that if $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for a vector space V. Then for each $\vec{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that $\vec{x} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n$

proof.

Let $\vec{x} \in V$ be given as above. that is $\vec{x} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n$

suppose $\vec{x}$ is also equal to $\vec{x} = d_1\vec{b}_1 + \ldots + d_n\vec{b}_n$ for scalars $d_1, \ldots, d_n$.

$\Rightarrow \vec{x} - \vec{x} = (c_1\vec{b}_1 + \ldots + c_n\vec{b}_n) - (d_1\vec{b}_1 + \ldots + d_n\vec{b}_n)$

$= (c_1 - d_1)\vec{b}_1 + \ldots + (c_n - d_n)\vec{b}_n$ homogeneous eqt.

$= \vec{0}$

$\Rightarrow$ since $\vec{b}_1, \ldots, \vec{b}_n$ are L.I. there are none trivial solut. to the homogenous equation

$c_1 - d_1 = 0 \Rightarrow c_1 = d_1$

$\vdots \qquad \vdots$

$c_n - d_n = 0 \Rightarrow c_n = d_n$

$\therefore$ The representation is unique.

4.) (5 pts) Find the coordinate vector $[\vec{x}]_B$ of $\begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$ relative to the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \right\}$

$\vec{x} \qquad \vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$

$\Rightarrow P_B = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & -2 \\ -3 & 9 & 4 \end{bmatrix}$

so $[\vec{x}]_B = P_B^{-1}\vec{x}$

$= \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}_B$

$\vec{x} \xleftarrow{P_B} [\vec{x}]_B$

5.) (5 pts) The set $B = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for P_2. Find the coordinate vector of $p(t) = 3 + t - 6t^2$ relative to B.

$P_2 \longleftrightarrow \mathbb{R}^3$

$1 - t^2 \longleftrightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$t - t^2 \longleftrightarrow \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$2 - 2t + t^2 \longleftrightarrow \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$P_B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$

so $[\vec{x}]_B = P_B^{-1}\vec{x}$

$= \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$

or $p(t) = 7(1 - t^2) - 3(t - t^2) - 2(2 - 2t + t^2)$