***Double Integrals over General Regions***

* **Double Integrals over General Regions**

Before we get into the “why” let’s do the following example to get a taste for the “how”:

**Example 1**: Evaluate:



In the last section we saw how to use double integrals to find the volume bounded between a multi- variable function and a rectangular region *R* on the *xy*-plane.

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| --- | --- |
| In this section we will look at more general regions that are not rectangles. We will label these more general regions *D.*  To do this we will suppose that  is a bounded region and can be enclosed in a rectangular region .  Frogmanipulate 16.09 |  |
| From here, the process is very similar. We will divide  into small rectangles (note that on the edge we have non-rectangular shapes which is okay since we are approximating). We then find the volume of each rectangular cylinder and add them to each other.  Crocodile manipulate 16.10 |  |
| To find a better approximation we will increase the number of the rectangles to infinity which takes us to the double integral definition of volume: |  |

Plane regions can be extremely complex, and the theory of double integrals over very general region is a topic for advanced courses in mathematics. We will limit our study of double integrals to two basic types of regions:

|  |  |
| --- | --- |
| A **Type  region** is bounded on the left and right by vertical lines  and  and is bounded below and above by the continuous curves  and . Then:    where |  |

Since  is fixed for the first integration, draw a vertical line in the region. The lower point of the intersection is the curve  and the higher point is . These are the lower and upper *y*-limits of integration.

|  |  |
| --- | --- |
| A **Type  region** is bounded below and above by horizontal lines  and  and is bounded on the left and right by the continuous curves  and . Then:    where |  |

Since  is fixed for the first integration, draw a horizontal line in the region. The leftmost point of the intersection is the curve  and the rightmost point is . These are the lower and upper *x*-limits of integration.

**Example 2**: Evaluate  over the region enclosed between , ,  and .

|  |  |
| --- | --- |
|  | Turtle  Manipulate  16.13 and 16.14 |

**Example 3**: Evaluate  over the region enclosed between ,  and .

|  |  |
| --- | --- |
|  | Rabbit  Manipulate  16.16 and 16.17 |

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| --- |
| **Historical Note**: Our definition of the definite integral is based upon the work of the German mathematician Bernhard Riemann (1826 – 1866). The son of a pastor, Riemann’s plan was to become a minister like his father. However, he was too good at math to set it aside. So when he arrived at university, and with his family’s permission, he began to study under Carl Gauss. His work as one of the key contributors to the rigorization of mathematics remains very influential.  Personally, he was shy and suffered from numerous nervous breakdowns. He had a terrible fear of public speaking. He was also a perfectionist who wouldn’t publish anything unless he felt it was perfect. The story goes that some of his unpublished papers were discarded by a maid who did not realize their potential value. He died in Italy having had to flee war in Germany.  While he did not pursue the ministry, Riemann remained dedicated to his faith for his lifetime. He saw his work as a mathematician as another way to serve God and considered his faith the most important aspect of his life. His tombstone ends with the inscription, “For those who love God, all things must work together for the best.” |

**Example 4**: Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane .

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| --- | --- |
|  | Dog  Manipulate  16.19 and 16.20 |

**Example 5**: Find the volume of the solid bounded by the cylinder  and the planes  and .

**Example 6**: Evaluate 

* **Properties of Double Integrals**

The following properties can be proven just like we did in previous calculus classes:

* 
*  where  is constant

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. For this purpose, we consider the solid consisting of the points between the plane  and the region , in the *xy*-plane. The volume of this solid is

|  |  |  |
| --- | --- | --- |
|  |  | Cat  Manipulate 16.25 |

The solid has volume:



Therefore:



**Example 7**: Use a double integral to find the area of the region enclosed between the parabola  and the line .