***Arc Length and Curvature***

* **Length of a curve**

We defined the length of a parametric curve in two dimensions as provided that the various functions and parameters are all well-behaved.

The length in three dimensions is defined similarly: Suppose  on where  and  are continuous. If the curve is traveled exactly once as  increases from  to , then:



Notice that we can also write: 

**Example 1**: A glider is soaring upward along the helix . How long is the glider’s path from  to ?

* **Arc Length**

The length of a curve is a constant (a number). The arc length is similar, but a function.



Notice that 

* **Curvature**

A parametrization  is called **smooth** on an interval *I* if  is continuous and non-zero on *I*. A curve is called **smooth** if it has a smooth parameterization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If *C* is a smooth curve defined by the vector function , recall that the unit tangent vector  is given by  and indicates the direction of the curve.

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| In the picture, you can see that  changes directions very slowly when *C* is fairly straight, but it changes direction more quickly when *C* bends or twists sharply. The curvature  at a given point is a measure of how quickly the curve changes direction at a point. |  |

 (curvature definition #1)

While possible, it is a nuisance to find  as a function of *s*. But notice that with the chain rule, we have . We can solve for  and  and thus:

 (curvature definition #2)

**Example 2**: Find the curvature of a circle with radius , centered at the origin, on the *xy*-plane.

Although the previous formulas work for finding the curvature, the following formula is more convenient to apply.

(curvature definition #3)

Note: This formula is not intuitive and its derivation requires (1.) using the product rule, (2.) the fact that , and (3.) knowing that  implies . The full derivation is in the text.

**Example 3**: Find the curvature of a straight line parametrized by .

**Example 4**: Find the curvature of  at the point .

* **The Normal and Binormal Vectors**

At a given point on a smooth space curve , there are many vectors that are orthogonal to the unit tangent vector . We’ll show that  is one of them!

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| **Historical note**: The mathematicians Jean Frederic Frenet (1847) and Joseph Alfred Serret (1851) independently discovered and described the kinematic properties of a particle moving along a curve using the tangent, normal, and binormal vectors. However, our modern notation for vectors and linear algebra did not exist for them. Today, their formulas are called the Frenet-Serret formulas and relate what we now call the TNB-frame and the curvature and torsion . |

Since  is not a generally a unit vector, we will define the **principal unit normal vector**  (or simply **unit normal**) as: . Note that  gives the direction of motion and  points in the direction the curve is turning. The vector  is called the **binormal vector** and it is perpendicular to  and .

The plane determined by the normal and binormal vectors  and  at a point *P* on a curve *C* is called the **normal plane** of *C* at *P*. It consists of all lines that are orthogonal to the tangent vector .

The plane determined by the vectors  and  is called the **osculating plane** of *C* at *P*. The name comes from the Latin *osculum*, meaning “kiss.”It is the plane that comes closest to containing the part of the curve near *P*.

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| http://www.cs.sjsu.edu/faculty/rucker/kaptaudoc/Image4.gif | File:Frenet trihedron.svg |  |

Note: For a plane curve, the osculating plane is simply the plane that contains the curve.

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| The circle that lies in the osculating plane of *C* at *P*, has the same tangent as *C* at *P*, lies on the concave side of *C* (toward which points), and has radius  and is culled the **osculating circle** of *C* at *P*. It is the circle that best describes how *C* behaves near *P*; it shares the same tangent, normal, and curvature at *P*. | http://www.mhhe.com/math/calc/smithminton2e/cd/folder_structure/text/chap11/section05/figure_1126.gif |

Nontechnical description: Imagine a car moving along a curved road on a vast flat plane. Suddenly, at one point along the road, the steering wheel locks in its present position. Thereafter, the car moves in a circle that "kisses" the road at the point of locking. The [curvature](https://en.wikipedia.org/wiki/Curvature) of the circle is equal to that of the road at that point. That circle is the osculating circle of the road curve at that point.

**Example 5**: Consider the space curve  from example 1

1. Find the unit normal vector
2. Find the binormal vector at 
3. equation of the normal plane at 
4. equation of the osculating plane at 
5. equation of the osculating circle at 