Assessment 9
Dusty Wilson
Math 220

Name: Key

A mathematician is a machine for turning coffee into theorems.

Paul Erdős
1913 - 1996 (Hungarian mathematician)

No work = no credit
Non CAS Calculators allowed

Warm-ups (1 pt each): Suppose $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\|\vec{u}\|$: 5 Unit vector in the direction of $\vec{u}$: $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

1.) (1 pt) Paraphrase the quote by Erdős (pronounced Air-Dersh). What was his point?

2.) (4 pts) Determine if the vectors $\vec{u} = \begin{bmatrix} -5 \\ 4 \\ 5 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -6 \end{bmatrix}$ are orthogonal. Explain.

$\vec{u} \cdot \vec{v} = -5 - 32 + 75 + 0 \neq 0$

$\vec{u}$ is <u>not</u> orthogonal to $\vec{v}$ as $\vec{u} \cdot \vec{v} \neq 0$

3.) (4 pts) Show that $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}} \right\}$ is an orthogonal basis for $\mathbb{R}^3$ and then express $\vec{x} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ as a linear combination of the orthogonal basis.

<u>1st</u>: show this is an orthogonal basis.

$\vec{v}_1 \cdot \vec{v}_2 = 10 - 10 + 0 = 0$ ✓

$\vec{v}_1 \cdot \vec{v}_3 = 5 - 5 + 0 = 0$ ✓

$\vec{v}_2 \cdot \vec{v}_3 = 2 + 2 - 4 = 0$ ✓

<u>2nd</u>: express $\vec{x}$ in terms of the basis,

$\dfrac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \dfrac{35}{50} = \dfrac{7}{10}$

$\dfrac{\vec{x} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \dfrac{7}{18}$

$\dfrac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \dfrac{5}{9}$

$\therefore \vec{x} = \frac{7}{10}\vec{v}_1 + \frac{5}{9}\vec{v}_2 + \frac{7}{18}\vec{v}_3$

4.) (4 pts) Find the orthogonal projection of $\vec{y} = \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}$ onto the subspace of $\mathbb{R}^3$ with orthogonal basis $\left\{ \overset{\vec{v}_1}{\begin{bmatrix} 7 \\ -1 \\ -2 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}} \right\}$.

$$\hat{y} = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \frac{-18}{54}\begin{bmatrix} 7 \\ -1 \\ -2 \end{bmatrix} + \frac{24}{18}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix} \quad (\vec{y} \text{ lives in the subspace})$$

5.) (4 pts) The vectors $\left\{ \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ -4 \end{bmatrix} \right\}$ are a basis for subspace W. Use the Gram-Schmidt process to produce an orthonormal basis for W.

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} \text{ and } \|\vec{v}_1\| = \sqrt{80}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \begin{bmatrix} 6 \\ 7 \\ -4 \end{bmatrix} - \frac{40}{80}\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -6 \end{bmatrix} \text{ and } \|\vec{v}_2\| = 9$$

the orthonormal basis is: $\frac{1}{\sqrt{80}}\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

6.) (4 pts) Use the Gram-Schmidt process to produce an orthogonal basis for the column space of matrix $A = \begin{bmatrix} -1 & 5 & 5 \\ 2 & -6 & 4 \\ 1 & -1 & 7 \\ 1 & -3 & -3 \end{bmatrix}$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{array}$$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \begin{bmatrix} 5 \\ -6 \\ -1 \\ -3 \end{bmatrix} - \frac{-21}{7}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}v_1 - \frac{\vec{x}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \begin{bmatrix} 5 \\ 4 \\ 7 \\ -3 \end{bmatrix} - \frac{7}{7}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{24}{8}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -4 \end{bmatrix}$$