**2.4: The Precise Definition of the Limit**

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| http://www-groups.dcs.st-andrews.ac.uk/%7Ehistory/BigPictures/Cauchy_4.jpeg | | Augustin Louis Cauchy 1789 - 1857 Augustin-Louis Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics. |
| http://www-groups.dcs.st-andrews.ac.uk/%7Ehistory/BigPictures/Weierstrass_2.jpeg | Karl Theodor Wilhelm Weierstrass 1815 - 1897  Karl Weierstrass is best known for his construction of the theory of complex functions by means of power series. | |

Definition: Let *f* be a function defined on some open interval that contains , except possibly at *a* itself. Then we say that the limit of  as *x* approaches *a* is *L* and we write   
if for all  there exists a  such that if  then .

Notation:

1.  is the lower case Greek letter \_\_\_\_\_\_\_\_\_\_\_\_ and refers to a small change in \_\_\_\_\_\_\_\_\_\_\_\_.
2.  is the lower case Greek letter \_\_\_\_\_\_\_\_\_\_\_\_ and refers to a small change in \_\_\_\_\_\_\_\_\_\_\_\_.

Generally, a proof has two parts:

1. \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_
2. \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

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**Example 1**: Prove 

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| 1. Guess | Back of the envelope calculation  For (a.), start with (2.) and end with (1.) |
| 1. The proof. | For (b.), start with (1.) and end with (2.) |

Example 1 revisited:

Two more general  diagrams. Remember, we choose  and then use it to determine a corresponding .

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| http://2.bp.blogspot.com/-YKjMrAbZlos/TXP_ime-rWI/AAAAAAAAL6Y/lQiQOANOqW4/s400/epsilon%2Bdelta%2Blimit.jpg | https://www.math.ucdavis.edu/%7Ekouba/CalcOneDIRECTORY/preciselimdirectory/precise.gif |
| In this diagram, the emphasis is on the fact that represents the width of an interval along the *x*-axis while  represents the height of an interval along the *y*-axis | In this diagram, the emphasis is on the fact that are the vertical boundary lines of a region while  are the horizontal boundaries. |

Recall: We say that the limit of  as *x* approaches *a* is *L* and we write   
if for all  there exists a  such that if  then .

Example 2: Prove 

|  |  |
| --- | --- |
| a) Guess | b) Prove it. |

We can use the precise definition to prove the validity of the limit laws in the previous section. The proof of the sum law is given in the text.

Example 3: Prove that if  and , then  provided the limits exist. This is called the difference law.

1. Guess 

We want .

We can do this, by requiring  and .

Since limits (1.) and (2.) exist, given  there exists a  such that if  then  and there exists a  such that if  then .

So we will choose 

1. Prove it.

Note: The previous exercise makes use of the **triangle inequality** which states 

Definition (an infinite limit): Let *f* be a function defined on some open interval that contains  (except possibly *a* itself). Then   
 if for all  there exists  such that if  then .

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| In the diagram, notice that when the *x*-values are within  of *a*, then the *y*-values are all above *M*. | http://tutorial.math.lamar.edu/Classes/CalcI/DefnOfLimit_files/image002.gif |

Example 4 (if time permits): Prove 

Additional Resources:

There is a nice series of lectures on the precise definition of the limit at the Khan Academy. The first is at: <http://www.khanacademy.org/video/limit-intuition-review>

For more, there were over 300 hits on YouTube for, “precise definition of a limit quadratic”