$x \rightarrow$ THE MACHINE f $\rightarrow$ y

Limits

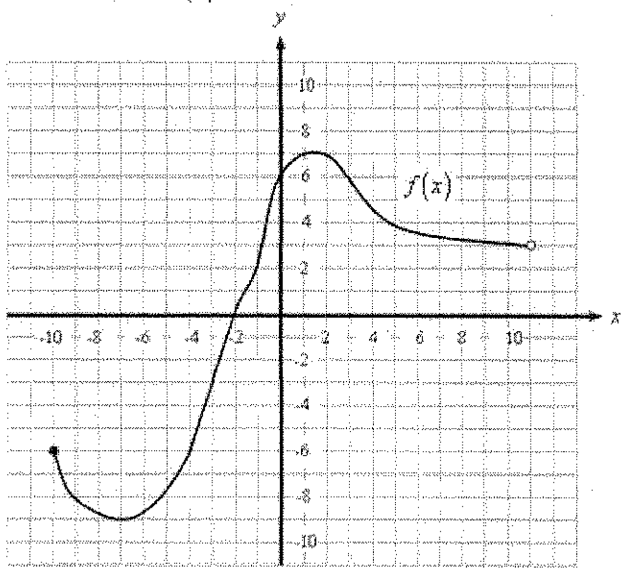
one input x has one & only one output y.

Section 9.1

Part 1: A Brief Review of Functions

■ Example 1: Functions and Graphs

Consider the complete graph of $f(x)$ that is given below.



$y = f(x)$

↑ output ↑ input

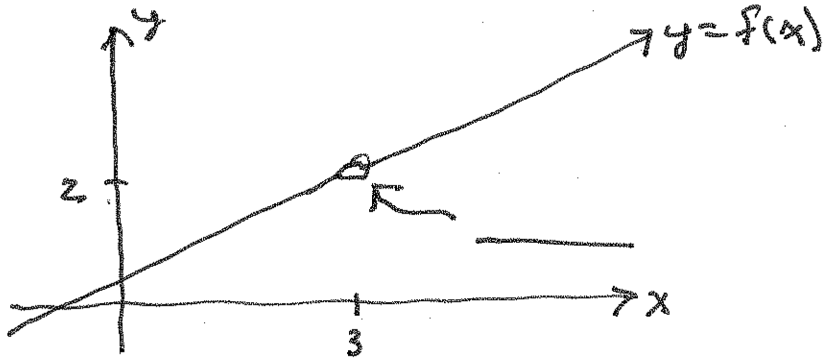
Use the graph to answer the following questions.

a.) $f(-10) =$ ______	b.) $f(-7) =$ ______
c.) $f(-2) =$ ______	d.) $f(0) =$ ______
e.) $f(3) =$ ______	f.) $f(11) =$ ______
g.) The domain of $f(x)$: $-10 \le x < 11$ (left to right)	
h.) The range of $f(x)$: $-9 \le y \le 7$ (bottom to top)	

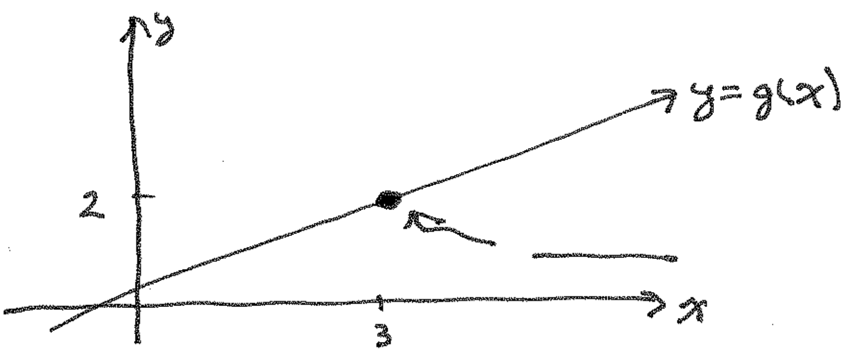
Part 2: Graphical Limits

▪ Example 2: The first two pictures

(a.)



(b.)



▪ Definition: The Limit

Let $f(x)$ be a function defined on an open interval containing c, except possibly at $x = c$. Then

$\lim_{x \to c} f(x) = L$

if we can make values of $f(x)$ as close to L as we desire by choosing values of x sufficiently close to c.

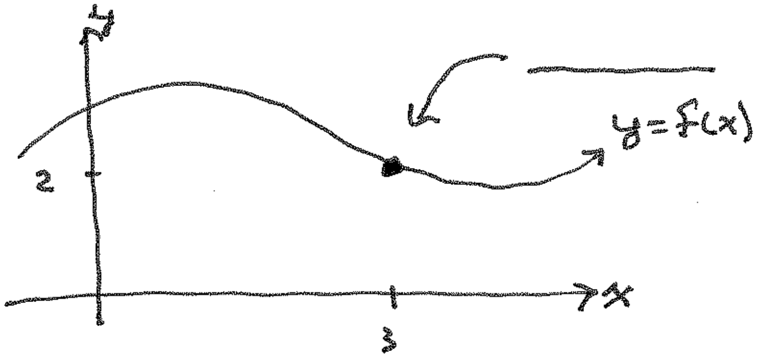
If the values of $f(x)$ do not approach a single finite L, the limit does not exist.

Notation: DNE means, "Does not exist."

Notation: We read $\lim_{x \to c} f(x) = L$ as, "The limit as x approaches c of $f(x)$ is L."

■ Example 3: Evaluating functions vs. evaluating limits

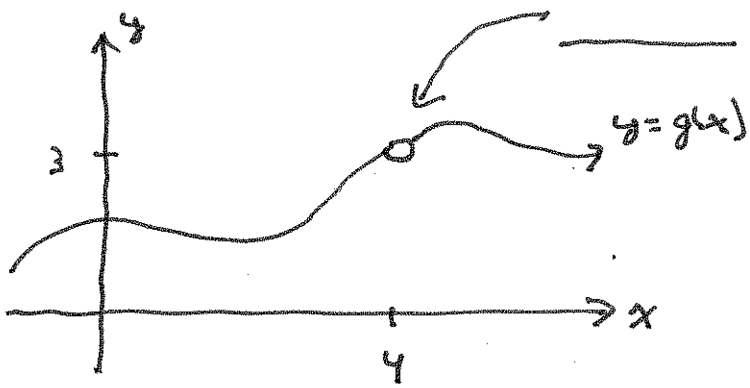
(a.)



(i) $f(3) =$ ______

(ii) $\lim_{x \to 3} f(x) =$ ______

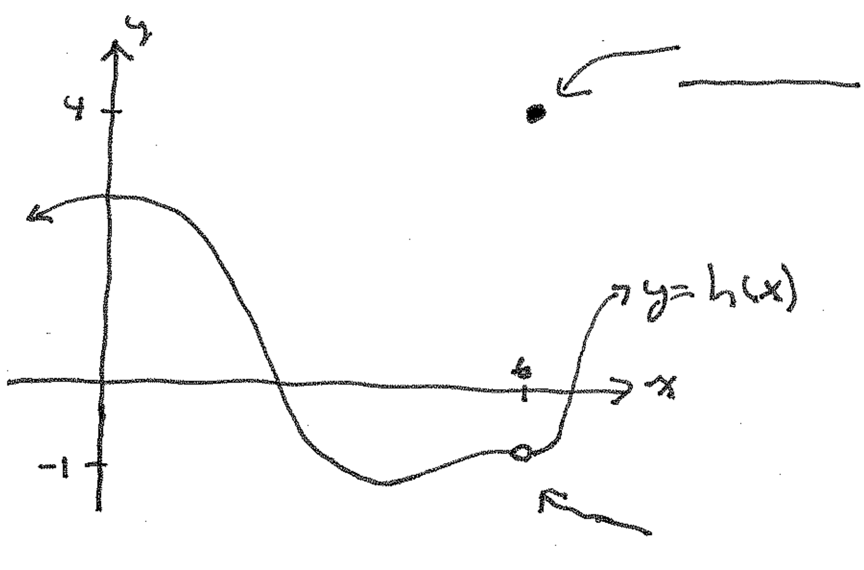
(b.)



(i) $g(4) =$ ______

(ii) $\lim_{x \to 4} g(x) =$ ______

(c.)

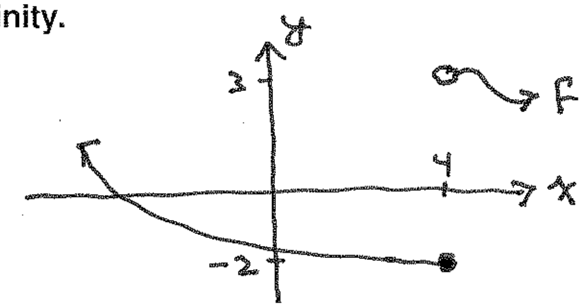


(i) $h(6) =$ ______

(ii) $\lim_{x \to 6} h(x) =$ ______

- **Example 4: Evaluating functions, left-hand and right-hand limits, limits, and limits at infinity.**

(a.)



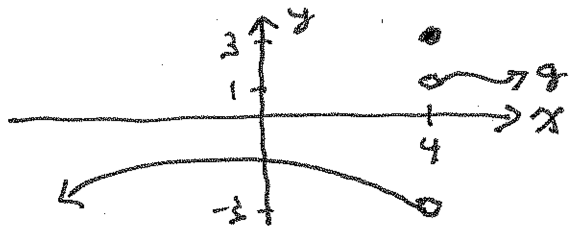
(i) $f(4)$

(ii) $\lim_{x \to 4^+} f(x)$

(iii) $\lim_{x \to 4^-} f(x)$

(iv) $\lim_{x \to 4} f(x)$

(b.)



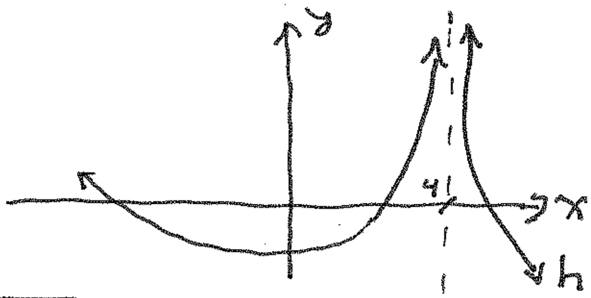
(i) $g(4)$

(ii) $\lim_{x \to 4^+} g(x)$

(iii) $\lim_{x \to 4^-} g(x)$

(iv) $\lim_{x \to 4} g(x)$

(c.)



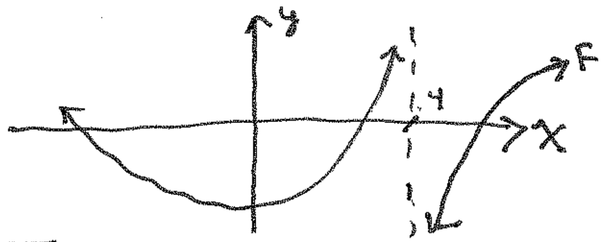
(i)

(ii)

(iii)

(iv)

(d.)



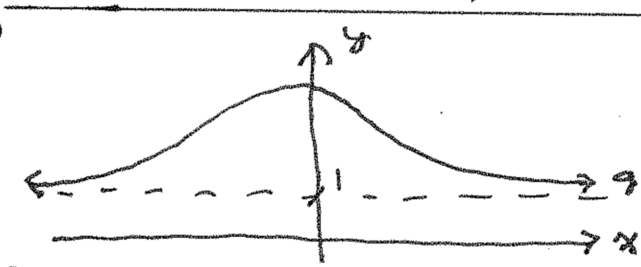
(i)

(ii)

(iii)

(iv)

(e.)



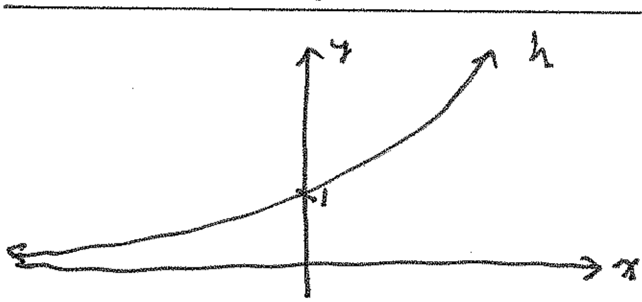
(i)

(ii)

(iii)

(iv)

(f.)



(i)

(ii)

(iii)

(iv)

Q: When is the function defined?

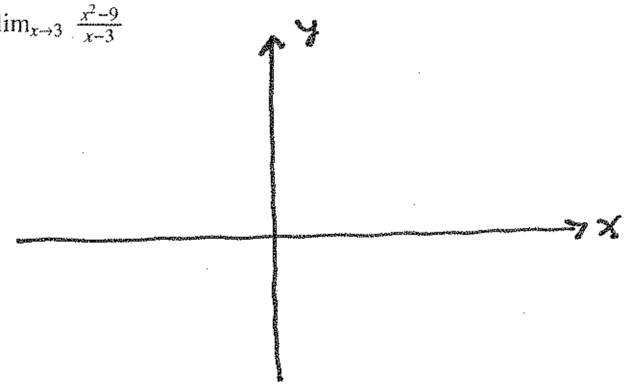
Q: When is the limit defined?

Part 3: Limits Algebraically

y1= (x^2-9)/(x-3)

▪ Example 5: Evaluate graphically

$\lim_{x\to 3} \frac{x^2-9}{x-3}$



Q: why the parenthesis?

Try the window :)

$-9.4 \le x \le 9.4$

$-9.4 \le y \le 9.4$

▪ Example 6: Evaluate graphically

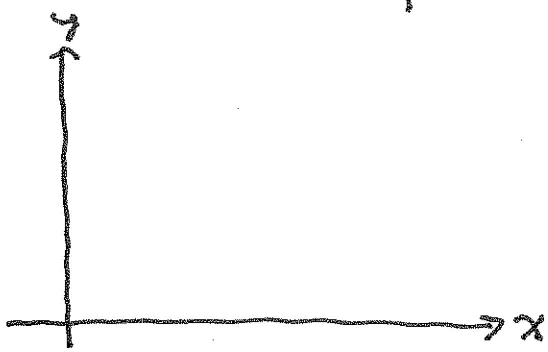
$\lim_{x\to -1} (4x^3 - 2x^2 + 2)$

A piecewise defined function

$y1 = (10-2x)(x < 3) + (x\wedge 2 - 6x + 15)(x \geq 3)$

Example 7: Evaluate graphically

$\lim_{x\to 3} f(x)$ where $f(x) = \begin{cases} 10 - 2x, & x < 3 \\ x^2 - 6x + 15, & x \geq 3 \end{cases}$



Hint: $<$ and $\geq$ are in the _____ menu.

A nice window is:

____ $\leq x \leq$ ____

____ $\leq y \leq$ ____

Properties of Limits

If k is a constant, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then

I. $\lim_{x\to c} k = k$
II. $\lim_{x\to c} x = c$
III. $\lim_{x\to c} [f(x) \pm g(x)] = L \pm M$
IV. $\lim_{x\to c} [(f \cdot g)(x)] = L \cdot M$ ← Notice $\lim_{x\to c} x^2 = \lim_{x\to c}(x \cdot x) = c \cdot c = c^2$
V. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
VI. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ provided that $L > 0$ when n is even.

Notation: "IFF" means "if and only if."

Definition: The Limit

$\lim_{x\to c} f(x) = L$ iff $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$

That is, the limit from the right must equal the limit from the left in order for the limit to exist.

Notice that the argument $4x^3 - 2x^2 + 2$ is a polynomial.
polynomials are our ______!

■ Example 6 revisited algebraically

$\lim_{x \to -1} (4x^3 - 2x^2 + 2)$

Hint: To evaluate "the limit of a polynomial" ______
______

■ Example 7 revisited algebraically

$\lim_{x \to 3} f(x)$ where $f(x) = \begin{cases} 10 - 2x, & x < 3 \\ x^2 - x, & x \geq 3 \end{cases}$.

■ Example 5 revisited algebraically

$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Can we find the limit algebraically?

continuous

Evaluating limits at $x = c$ when the function is ~~continuous~~ at $x = c$ is easy; simply evaluate the function at c.

▪ Example 8

$\lim_{x \to 4} \frac{x^2-16}{x-4}$

▪ Example 9

$\lim_{x \to 7} \frac{x^2-8x+7}{x^2-6x-7}$

Example 12

$\lim_{x \to -1} f(x)$ where $f(x) = \begin{cases} x^2 + \frac{4}{x}, & x \leq -1 \\ 3x^3 - x - 1, & x > -1 \end{cases}$

Q: can we find the limit
w/o the graph?

Example 13

Suppose that the cost C of removing p percent of the pollution from an industrial plant is modeled by:

$C(p) = \frac{730000}{100-p} - 7300$

a.) Find and interpret $\lim_{p \to 80} C(p)$

b.) Find and interpret $\lim_{p \to 100^-} C(p)$

c.) Can all the pollution be removed?

■ Example 10

$\lim_{x \to -3} \frac{x^2+6x+9}{x-2}$

■ Example 11

$\lim_{x \to 1} \frac{x^2-1}{x^2-2x+1}$

■ Summary of Examples 8 - 11

Evaluating limits of rational functions where the denominator approaches zero.

a.) If the numerator does not approach zero, then the limit D.N.E. (does not exist).

b.) If the numerator approaches zero, simplify and then try again.