Section 12.02

The Power Rule

Part 1: Differentials

The Differential: If $\frac{dy}{dx} = f'(x)$, then the differential $dy = f'(x)\,dx$.

Example 1: Find the differential dy if:

a.) $y = x^7 + 3x^2 + 2$

$$\frac{dy}{dx} = 7x^6 + 6x$$

$$\Rightarrow dy = (7x^6 + 6x)\,dx$$

b.) $y = x^3 e^x$

$$\frac{dy}{dx} = 3x^2 e^x + x^3 e^x$$

$$\Rightarrow dy = (3x^2 e^x + x^3 e^x)\,dx$$

Part 2: The Power Rule

Recall the power rule for derivatives: $\frac{d}{dx}[u(x)]^n = n[u(x)]^{n-1}\,u'(x)$. This leads to the power rule for integration where $\int n[u(x)]^{n-1}\,u'(x)\,dx = [u(x)]^n + C$.

Power rule for integration: Assuming that $n \neq -1$,

$$\int [u(x)]^n\, u'(x)\, dx = \frac{[u(x)]^{n+1}}{n+1} + C$$

or if $u = u(x)$, then

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$$

Example 2: $\int (3x^3+1)^4\, 9x^2\, dx \;=\; \int u^4\, du$

Let $u = 3x^3+1$

$\frac{du}{dx} = 9x^2$

$\Rightarrow du = 9x^2 dx$

$= \frac{1}{5} u^5 + c$

$= \frac{1}{5}(3x^3+1)^5 + c.$

Example 3: $\int (3x^2-4)^6\, x\, dx \;=\; \frac{1}{6}\int (3x^2-4)^6 \cdot 6x\, dx$

Let $u = 3x^2-4$

$\frac{du}{dx} = 6x$

$du = 6x\,dx$

$= \frac{1}{6}\int u^6\, du.$

$= \frac{1}{6}\cdot\frac{1}{7}u^7 + c$

$= \frac{1}{42}(3x^2-4)^7 + c$

Example 4: $\int \frac{x\,dx}{(x^2+1)^3}$ $= \frac{1}{2}\int \frac{2x\,dx}{(x^2+1)^3}$

Let $u = x^2+1$

$du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{u^3}$

$= \frac{1}{2}\int u^{-3}\,du$

$= \frac{1}{2}\frac{u^{-2}}{-2} + c$

$= -\frac{1}{4}(x^2+1)^{-2} + c$

Example 5: $\int 7x^3\sqrt{x^4+6}\,dx$ $= \frac{7}{4}\int 4x^3(x^4+6)^{1/2}\,dx$

Let $u = x^4+6$

$du = 4x^3dx$

$= \frac{7}{4}\int u^{1/2}\,du$

$= \frac{7}{4}\cdot\frac{u^{3/2}}{3/2} + c$

Example 6: $\int (x^2+1)^2\,dx$ $= \int (x^4+2x^2+1)\,dx$

substitution fails.

$= \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + c$

Let $u = x^2+1$

$du = 2x\,dx$

$\uparrow$

$*$

Example 7: $\int \frac{5\,x\,dx}{(x^2-1)^{13}}$ $= 5\int x\,(x^2-1)^{-13}\,dx$

Let $u = x^2 - 1$ $= \frac{5}{2}\int 2x(x^2-1)^{-13}\,dx$

$du = 2x\,dx$

$= \frac{5}{2}\int u^{-13}\,du$

$= \frac{5}{2}\,\frac{u^{-12}}{-12} + c$

$= -\frac{5}{24}(x^2-1)^{-12} + c$

Example 8: $\int \frac{x^3-1}{(x^4-4\,x)^3}\,dx$ $= \frac{1}{4}\int \frac{4(x^3-1)\,dx}{(x^4-4x)^3}$

Let $u = x^4 - 4x$

$du = (4x^3 - 4)\,dx$ $= \frac{1}{4}\int u^{-3}\,du$

$= 4(x^3-1)\,dx$

$= \frac{1}{4}\,\frac{u^{-2}}{-2} + c$

$= -\frac{1}{8}(x^4-4x)^{-3} + c$

Example 9: $\int \frac{x^2+1}{\sqrt{x^3+3\,x+10}}\,dx$ $= \frac{1}{3}\int 3(x^2+1)(x^3+3x+10)^{-1/2}\,dx$

Let $u = x^3 + 3x + 10$ $= \frac{1}{3}\int u^{-1/2}\,du$

$du = (3x^2 + 3)\,dx$

$= 3(x^2+1)\,dx$ $= \frac{1}{3}\,\frac{u^{1/2}}{1/2} + c$

$= \frac{2}{3}(x^3+3x+10)^{1/2} + c$

Part 3: Applications (time permitting)

Example 10: A new firm predicts that the number of franchises will grow at a rate $\frac{dn}{dt} = 9\sqrt{t+1}$ where t is in years, $0 \le t \le 10$. If there are presently three franchises (after zero years), how many franchises can be expected in eight years?

(1) $N(t) = \int 9\sqrt{t+1}\,dt$

$= \int 9(t+1)^{1/2}\,dt$ Let $u = t+1$, $du = dt$

$= \int 9u^{1/2}\,du$

$= 9 \cdot \frac{2}{3} u^{3/2} + c$

$= 6(t+1)^{3/2} + c$

(2) Find c.

$N(0) = 3 = 6 + c$

$\Rightarrow c = -3$

(3) $N(t) = 6(t+1)^{3/2} - 3$

and $N(8) = 6 \cdot 9^{3/2} - 3 = 159$

(4) we expect 159 franchises after 8 years.