

The Chain Rule

Part 1: Composition of functions

Example 1: If $f(x) = 3x^4$ and $g(x) = 3 - 2x$, find $h(x) = f(g(x))$.

$$h(x) = f(g(x)) = f(3 - 2x) = 3(3 - 2x)^4$$

Example 2: For the following functions h , find f and g such that $h(x) = f(g(x))$.

a.) $h(x) = (2x^4 - 5)^{25}$

$$g(x) = 2x^4 - 5$$

$$f(x) = x^{25}$$

b.) $h(x) = (3 - 2x)^{10}$

$$g(x) = 3 - 2x$$

$$f(x) = x^{10}$$

c.) $h(x) = \frac{2}{3}(x^6 + 3x^2 - 11)^8$

$$g(x) = x^6 + 3x^2 - 11$$

$$f(x) = \frac{2}{3}x^8$$

Example 3: For the following functions h , find f and g such that $h(x) = f(g(x))$.

a.) $h(x) = \frac{1}{3(3x^2+3x+5)^{3/4}}$

$$g(x) = 3x^2 + 3x + 5$$

$$f(x) = \frac{1}{3x^{3/4}}$$

b.) $h(x) = \sqrt{x^2 + 3x}$

$$g(x) = x^2 + 3x$$

$$f(x) = \sqrt{x}$$

Part 2: The Chain Rule

Derivative Rule: The chain rule

If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$

This can be memorized as, $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$.

Example 2 revisited: For each example, find $h'(x)$.

a.) $h(x) = (2x^4 - 5)^{25}$

$$h'(x) = 25(2x^4 - 5)^{24} \cdot 8x^3$$

b.) $h(x) = (3 - 2x)^{10}$

$$h'(x) = 10(3 - 2x)^9 \cdot (-2)$$

Example 2 revisited (continued from previous page)

$$c.) h(x) = \frac{2}{3} (x^6 + 3x^2 - 11)^8$$

$$h'(x) = \frac{16}{3} (x^6 + 3x^2 - 11)^7 (6x^5 + 6x)$$

Example 3 revisited: For each example, find $h'(x)$.

$$a.) h(x) = \frac{1}{3(3x^2 + 3x + 5)^{3/4}} = \frac{1}{3} (3x^2 + 3x + 5)^{-3/4}$$

$$h'(x) = -\frac{1}{4} (3x^2 + 3x + 5)^{-7/4} (6x + 3)$$

$$b.) h(x) = \sqrt{x^2 + 3x} = (x^2 + 3x)^{1/2}$$

$$h'(x) = \frac{1}{2} (x^2 + 3x)^{-1/2} \cdot (2x + 3)$$

Example 4: Find the tangent line to $y = (x^2 + 1)^3$ at $(2, 125)$.

$$y' = 3(x^2 + 1)^2 \cdot 2x$$

$$\text{@ } x = 2 : y' = 3 \cdot 5^2 \cdot 4 = \overset{1800}{\cancel{7500}}$$

$$\text{(line: } y - 125 = \overset{1800}{\cancel{7500}}(x - 2)$$

Example 5 (for you): Find the tangent line to $y = \left(\frac{1}{x^3-x}\right)^3$ at $x=2$.

$$y = (x^3 - x)^{-3} \quad \text{pt } \left(2, \frac{1}{216}\right)$$

$$y' = -3(x^3 - x)^{-4}$$

Example 6: Differentiate the following

a.) $y = \frac{5}{7}(2x^3 - x + 6)^{14}$

$$y' = 10(2x^3 - x + 6)^{13}(6x^2 - 1)$$

b.) $p = (q^3 + 1)^{-5}$

$$p' = -5(q^3 + 1)^{-6} \cdot 3q^2$$

c.) $f(x) = \frac{1}{(x^2+2)^3} = (x^2+2)^{-3}$

$$f'(x) = -3(x^2+2)^{-4} \cdot 2x$$

Example 7: Differentiate the following

$$\text{a.) } g(x) = \frac{1}{(2x^3 + 3x + 5)^{3/4}} = (2x^3 + 3x + 5)^{-3/4}$$

$$g'(x) = -\frac{3}{4} (2x^3 + 3x + 5)^{-7/4} \cdot (6x^2 + 3)$$

$$\text{b.) } y = \frac{(3x+1)^5 - 3x}{7}$$

$$y' = \frac{1}{7} \left[5(3x+1)^4 \cdot 3 - 3 \right]$$

Example 8: $R(x) = 15(3x+1)^{-1} + 5x - 15$ gives the the dollars of revenue from the sale of x items. Find and interpret $\overline{MR}(4)$.

$$\overline{MR} = -15(3x+1)^{-2} \cdot 3 + 5$$

$$\begin{aligned} \text{@ } x=4 \quad \overline{MR} &= \frac{-45}{169} + 5 \\ &= \frac{800}{169} \approx 4.73 \end{aligned}$$

The revenue from the sale of the 5th item (x goes from 4 to 5) is about \$4.73.

Quiz - Just for you

a.) Write down the product rule

$$(u \cdot v)' = u'v + v'u$$

b.) Write down the quotient rule

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

c.) Write down the chain rule

$$\text{If } h(x) = f(g(x)), \quad h'(x) = f'(g(x)) \cdot g'(x)$$

Example 9: The daily sales S attributed to an advertising campaign are given by the function $S(t) = 1 + \frac{3}{t+3} - \frac{18}{(t+3)^2}$ where t is the number of weeks the advertisement runs.

$$S(t) = 1 + 3(t+3)^{-1} - 18(t+3)^{-2}$$

a.) Find the ROC when $t = 8$.

$$S'(t) = -3(t+3)^{-2} \cdot 1 + 36(t+3)^{-3} \cdot 1$$

$$S'(8) = \frac{-3}{11^2} + \frac{36}{11^3} \approx 0.00225$$

b.) Find the ROC when $t = 10$.

$$S'(10) = \frac{-3}{13^2} + \frac{36}{13^3} \approx -0.00137$$

c.) Should the campaign continue after the tenth week?

Sales are decreasing... stop!