3.2a
1/4

Find all rational & irrational roots of

$P(x) = 2x^3 - 7x^2 + 4x + 3$. Exactly

c a factor of 2. b a factor of 3

$c: \pm 1, \pm 2$ $b: \pm 1, \pm 3$

$\frac{b}{c}: \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}$

1) Find possible ~~zeros~~ values of b & c.

2) List possible values of $\frac{b}{c}$.

3) use synthetic div. to find zeros.

4) list all zeros.

	2	−7	4	3
3	2	−1	1	6
$\frac{3}{2}$	2	−4	−2	0

$$P(x) = 2x^3 - 7x^2 + 4x + 3 = \underbrace{\left(x - \frac{3}{2}\right)}_{\text{root at } x = \frac{3}{2}}\underbrace{(2x^2 - 4x - 2)}_{\text{quadratic.}}$$

$$0 = 2x^2 - 4x - 2.$$

$$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4(2)(-2)}}{4} = \frac{4 \pm \sqrt{32}}{4} = 1 \pm \sqrt{2}.$$

roots at $x = \frac{3}{2}$ and $x = 1 \pm \sqrt{2}$.

3.2a
2/4

* We will refer back to this problem
— do on side board —

Ex2: Find all zeros exactly for $P(x) = 3x^3 - 10x^2 + 5x + 4$

$$P(x) = 3x^3 - 10x^2 + 5x + 4$$

c a factor of 3

b a factor of 4

c: ±1,

b: ±1, ±2, ±4

$$\frac{b}{c}: \pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$$

	3	−10	5	4
4	3	2	13	56
2	3	−4	−3	−2
$\frac{4}{3}$	3	−6	−3	0

$$P(x) = \left(x - \frac{4}{3}\right)\underbrace{(3x^2 - 6x - 3)}$$

$$x = \frac{6 \pm \sqrt{36 - 4(3)(-3)}}{2(3)}$$

$$= \frac{6 \pm \sqrt{72}}{6}$$

$$= 1 \pm \sqrt{2}$$

zeros : $x = \frac{4}{3}$, $x = 1 \pm \sqrt{2}$

$\boxed{\begin{array}{c}3.2b \\ 1/4\end{array}}$

Ex 1: $P(x) = \frac{1}{2}x^4 - \frac{11}{2}x^2 + 6x + 2$

$\Rightarrow \cancel{= \frac{1}{2}(x^4 - 11x^2} \; 6x$

$= \frac{1}{2}(x^4 - 11x^2 + 12x + 4).$

c a factor of 1 b a factor of 4.

c: ± 1 b: $\pm 1, \pm 2, \pm 4$

$\frac{b}{c}: \pm 1, \pm 2, \pm 4$

	1	0	-11	12	4
4	1	4	5	32	132
$\boxed{2}$	1	2	-7	-2	0

$P(x) = \frac{1}{2}(x-2)(x^3 + 2x^2 - 7x - 2)$

c a factor of 1 b a factor of -2

c: ± 1 b: $\pm 1, \pm 2.$

$\frac{b}{c}: \pm 1, \pm 2.$

	1	2	-7	-2
2	1	4	1	0

$P(x) = \frac{1}{2}(x-2)(x-2)(x^2 + 4x + 1)$

$= \frac{1}{2}(x-2)^2(x^2 - 4x + 1)$

$x^2 - 4x + 1 = 0$

$x = \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2}$

$= 2 \pm \sqrt{3}$

roots: $x = 2$ multiplicity 2

$x = 2 \pm \sqrt{3}.$

3.2
2/4

Factor Theorem

r is a root of the polynomial $P(x)$ iff $(x-r)$ is a factor of $P(x)$.

E e.g. $P(x) = (x - \frac{4}{3})(3x^2 - 6x - 3) \iff x = \frac{4}{3}$ is a root.

Fundamental Theorem of Algebra

Every polynomial w/complex coefficients w/ degree >0 has at least one root r_1 (r_1 is possibly complex).

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ $(n > 0)$.

FToA: $\exists\, r_1 \in \mathbb{Z}$ s.t. $P(r_1) = 0$. e.g. $x = \frac{4}{3}$

Factor Theorem: $P(x) = (x - r_1)\underbrace{Q_1(x)}_{\text{poly w/degree } n-1.}$

e.g. $3x^2 - 6x - 3$

By FToA: $Q_1(x)$ has 1 root r_2

$\Rightarrow Q_1(x) = (x - r_2)Q_2(x)$

$\Rightarrow P(x) = (x - r_1)(x - r_2)Q_2(x)$. degree $n-2$.

$\vdots$

$P(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$.

3.2b
3/4

We have just outlined the proof of:

$\boxed{N \text{ roots Theorem}}$

Every polynomial w/degree $N>0$ has exactly N roots (Not necessarily distinct and possibly complex).

Argh.

$$P(x) = 3x^3 - 10x^2 + 5x + 4 = 3\left(x - \tfrac{4}{3}\right)\left(x-(1+\sqrt{2})\right)\left(x-(1-\sqrt{2})\right)$$

Decent Stopping Point.

Definition: Rational Number.

a Number is rational if it can be represented as the quotient $\frac{b}{c}$ where $b, c \neq 0$ are integers.

Our GoAL: Find all Rational Roots $x = \frac{b}{c}$ of $P(x)$. We assume $\frac{b}{c}$ is reduced to lowest terms which is to say b & c are relatively prime.

Assume $a_N, a_{N-1}, \dots, a_1, a_0 \in \mathbb{Z}$.

If $x = \frac{b}{c}$ is a root of $P(x)$, then

$$a_N\left(\frac{b}{c}\right)^N + a_{N-1}\left(\frac{b}{c}\right)^{N-1} + \dots + a_1\left(\frac{b}{c}\right) + a_0 = 0$$

$$(*) \Rightarrow a_N b^N + a_{N-1} b^{N-1} c + \dots + a_1 b c^{N-1} + a_0 c^N = 0$$

$$\Rightarrow \quad a_N b^N = c\left(-a_{N-1}b^{N-1} - \dots - a_1 b c^{N-2} - a_0 c^{N-1}\right)$$

relatively prime since $\frac{b}{c}$ in lowest terms.

$\Rightarrow$ c is a factor of a_N.

$$(*) \quad a_N b^N + a_{N-1} b^{N-1} c + \dots + a_1 b c^{N-1} + a_0 c^N = 0$$

$$\Rightarrow \quad a_0 c^N = b\left(-a_N b^{N-1} - a_{N-1} b^{N-2} c - \dots - a_1 c^{N-2}\right)$$

relatively prime.

$\Rightarrow$ b is a factor of a_0.

This proves the rational root theorem.

Rational Root Theorem

If the rational number $x = \frac{b}{c}$ is a root of $P(x) = a_N x^N + \dots + a_1 x + a_0$ w/ integer coeffients, then b is an integer factor of a_0 and c is an integer factor of a_N

Ex 2: $\quad P(x) = x^3 + 5x^2 - 2x - 24$

Ex 3: $\quad P(x) = 3x^5 - 2x^4 + 6x^3 + 20x^2 - x - 10.$

Dusty Wilson
Math 115 – Fall 2002
Root Finding Handout: Homework for section 3.3

Instructions:
All of the following polynomials have roots that can be expressed in exact form. Find all the roots (rational, irrational, and complex roots) of these polynomials, along with the multiplicities if the root if the root has a multiplicity greater than one. Do not give any decimal approximations to the answers!

Your graphing calculator, the Rational Root's Theorem (Theorem 6 on page 231), and synthetic division should be used to help determine your answers.

Note: The program on page 245 may be of interest to you.

1.) $P(x) = 7x^3 + 10x^2 - 11x - 6$

2.) $P(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$

3.) $P(x) = x^4 + 3x^3 - 10x^2 + 7x - 1$

4.) $P(x) = 72x^3 - 66x^2 - 193x + 195$

5.) $P(x) = 3x^6 - 5x^5 + 2x^4 + 3x^2 - 5x + 2$

6.) $P(x) = 11x^5 + 39x^4 + 46x^3 + 14x^2 - 9x - 5$

3.3
1/3

3.3 : Approx Real Zeros.

Ex 1: Let $P(x) = x^3 - 8x^2 + 15x - 2$. Use synthetic division to locate roots between successive integers.

How many real roots $\longrightarrow$ at least 1.

	1	-8	15	-2	
0				-2	root
1				6	
2				4	root
3				-2	
4				-6	
5				-2	root.
6				16	

$\boxed{\begin{array}{c} 3.3 \\ 2/b \end{array}}$

Bounding Real Roots

Given an n^{th} degree poly w/real coefficients, $n>0$, $a_n>0$, and $P(x)$ divided by $(x-r)$ w/syndiv.

$r>0$ is an <u>upper bound</u> if all entries are non-neg.

$r<0$ is a <u>lower bound</u> if all entries alternate sign.

Ex 2: Let $P(x) = x^4 - 5x^3 - x^2 + 40x - 70$. Use syndiv to bound <u>ALL</u> real roots between successive integers.

	1	−5	−1	40	−70	
0					−70	
1					−35	
2					−18	
3					−13	$\}$ zero.
4					10	
5					105	
6	1	1	5	70	350	UB

	1	−5	−1	40	−70	
−1					−105	
−2					−98	$\}$ zero
−3	1	−8	23	−29	17	LB

~~We have proven that there are only 2 real roots.~~

$\boxed{\dfrac{33}{5/3}}$

Ex 3: Use bisection to estimate the zero of

$P(x) = x^4 - 5x^3 - x^2 + 40x - 70$ on the interval $[3, 4]$

Ex 4: Let $P(x) = x^3 - 32x^2 - 2921x + 2952$.

a) Use syn div. to bound roots.

b) find roots w/calc.

$\boxed{\begin{array}{c} 3.4a \\ 1/2 \end{array}}$

Rational Functions

$f(x) = \dfrac{1}{x}$ Domain.

$g(x) = \dfrac{1}{x+2}$. vertical asymptote.

$h(x) = \dfrac{-1}{x+2}$ horizontal asymptote. ← Non-vertical

$f(x) = \dfrac{1}{x(x+2)}$. sign diagram.

$g(x) = \dfrac{x-1}{x(x+2)}$. root.

$h(x) = \dfrac{(x+1)(x-1)}{x(x+2)}$ horizontal asymptote. ← Non-vertical.

$f(x) = \dfrac{x}{x}$ hole.

$g(x) = \dfrac{(x+1)(x-1)^2}{x(x+2)(x-1)}$ hole