Assessment 7
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Math 220

No work = no credit

Name: Key

I mean the word proof not in the sense of the lawyers, who set two half proofs equal to a whole one, but in the sense of a mathematician, where half proof = 0, and it is demanded for proof that every doubt becomes impossible.

Carl Gauss
1777 - 1855 (German mathematician)

Find all eigenvalues (3 pts): $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$: 1, 2 $\quad \begin{bmatrix}5 & 5\\5 & 5\end{bmatrix}$: 0, 10 $\quad \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$: 1, 0

1.) (1 pt) What did Gauss mean by the word, "proof"? Answer using complete English sentences.

In math, proof removes ever possible doubt or question.

2.) (5 pts) Diagonalize $A=\begin{bmatrix}2 & 3\\4 & 1\end{bmatrix}$, if possible.

① solve $0 = \begin{vmatrix}2-\lambda & 3\\4 & 1-\lambda\end{vmatrix} = (2-\lambda)(1-\lambda)-12 = \lambda^2-3\lambda-10$

$\Rightarrow 0 = (\lambda-5)(\lambda+2)$ so $\lambda = 5, -2$

② $\lambda=5$: $A-5I \sim \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \Rightarrow$ eigenvec: $\begin{bmatrix}1\\1\end{bmatrix}$

$\lambda=-2$: $A+2I \sim \begin{bmatrix}1 & 0.75\\0 & 0\end{bmatrix} \Rightarrow$ eigenvec: $\begin{bmatrix}3\\-4\end{bmatrix}$

③

$$A = \begin{bmatrix}1 & 3\\1 & -4\end{bmatrix}\begin{bmatrix}5 & 0\\0 & -2\end{bmatrix}\left(-\frac{1}{7}\right)\begin{bmatrix}-4 & -3\\-1 & 1\end{bmatrix}$$

3.) (2 pts) How can diagonalization help us find powers of matrices?

If $A = PDP^{-1}$ w/ D diagonal), then $A^t = PD^tP^{-1}$ and D^t is easy to calculate because we just need powers of the diagonal entries.

4.) (5 pts) The matrix $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ has the eigenvalue $\lambda = 4$. Find of a basis for the corresponding eigenspace. and its dimension.

$$A - 4I \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{eigenvecs } \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The dimension of $E_{\lambda=4}$ is 2.

5.) (4 pts) Find the eigenvalues of $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$. Please show your algebraic work (but using tech as an aid is okay).

solve $0 = \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix}$

$= (1-\lambda) \begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix}$

$= (1-\lambda)\left[(5-\lambda)(-2-\lambda) - 18\right]$

$= (1-\lambda)\left[\lambda^2 - 3\lambda - 28\right]$

$\Rightarrow 0 = (1-\lambda)(\lambda-7)(\lambda+4)$

$\Rightarrow \lambda = 1, 7, -4.$

6.) (5 pts) Prove that if $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

proof

Suppose A & B are similar i.e., there is an invertible P s.t. $A = PBP^{-1}$.

$\Rightarrow A - \lambda I = PBP^{-1} - \lambda I = PBP^{-1} - P(\lambda I)P^{-1}$

$= P(B - \lambda I)P^{-1}$

$\Rightarrow \det(A - \lambda I) = \det(P(B-\lambda I)P^{-1}) = \det(P)\det(B-\lambda I)\det(P^{-1})$.

And $\det(P)\det(P^{-1}) = 1$

$\therefore \det(A-\lambda I) = \det(B - \lambda I)$. Q.E.D.