Assessment 6
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Math 220

Name: Key

As for everything else, so for a mathematical theory: beauty can be perceived but not explained.

Arthur Cayley
1821 - 1895 (English mathematician)

No work = no credit

Warm-ups (1 pt each):
Note: Assume $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$

$\vec{e}_2 \vec{e}_2^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \vec{e}_2^T \vec{e}_2 = [1]$ $\quad \vec{e}_2^T \vec{e}_1 = [0]$

1.) (1 pt) The quote above is by Cayley who was one of the founders of linear algebra. According to Cayley, how do we explain beauty in mathematics? Answer using complete English sentences.

Cayley felt that beauty in all areas (including math) could only be perceived (not explained).

2.) (5 pts) Let $B = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ are bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from B to C **and** the change-of-coordinates matrix from C to B. Please clearly indicate which is which.

$\underset{C \leftarrow B}{P}$: $\left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{array}\right] \to \underset{C \leftarrow B}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$

$\underset{B \leftarrow C}{P} = \underset{C \leftarrow B}{P}^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}$

3.) (5 pts) In P_2, find the change-of-coordinates matrix from the basis $B = \{1-3t^2, 2+t-5t^2, 1+2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in B.

$B \xleftrightarrow{\mathbb{R}^3} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

so $[\vec{x}]_B = P_B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$P_B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$

AND $t^2 = 3(1-3t^2) - 2(2+t-5t^2) + 1(1+2t)$

$\vec{x} \xleftarrow{P_B} [\vec{x}]_B$

4.) (3 pts) Consider $A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix}$. Find the rank of A and the dimension of the null space of A What is their sum?

$A \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Rank = 3

nullity = 2

rank + nullity = 5

5.) (2 pts) If A is a 20 x 23 matrix with a five-dimensional null space, what is the rank of A? Why?

rank = 23 - 5 = 18 since rank + nullity = # of cols in A.

6.) (5 pts) Prove the Basis Theorem which states that if V is a p-dimensional vector space, $p \geq 1$ then: Any linearly independent set of exactly p elements in V is automatically a basis for V. And, any set of exactly p elements that spans V is automatically a basis for V.

proof.

(1) We previously showed (Thm 11) that a LI set S of p elements can be extended to a basis for V. But that basis must contain exactly p elements, since dim (V) = p. So S must already be a basis for V.

(2) Now suppose that S has p elements and spans V. Since V is non-zero, the spanning set thm implies that a subset S' of S is a basis of V. Since dim V = p, S' must contain p vectors. Hence S = S'.

Q.E.D.