Chapter 3: Fair-Division Games Math 107

Fair Shares – 1/nth n people, everyone gets 1/nth of the total

Fair Divisions –

Necessary Assumptions:

Rationality - fair & rational

Cooperation - work together

Privacy - don't know each others preferences

Symmetry - everyone is even

Continuous – can be divided "infinitely" many ways (cake, land, money)

Discrete – whole items not easily broken apart (cars, paintings, houses,)

Ex 1: Given the Fair-Division:

	s_1	s_2	s_3	s_4
Angie	22%	26%	28%	24%
Bev	25%	26%	22%	27%
Ceci	20%	30%	27%	23%
Dina	25%	25%	25%	25%

row total 100%

a) Who was the Divider?

Dina

b) What are "Fair-Shares" for Angie?

s_2 & s_3

c) What are "Fair-Shares" for Bev?

s_1, s_2, s_4

Divider-Chooser:

"You cut, I choose!"

The divider-chooser method goes all the way back to the Old Testament: When Lot and Abraham argued over grazing rights, Abraham proposed, "Let us divide the land into left and right. If you go left, I will go right; and if you go right, I will go left" (Genesis 13:1–9).

Exercises 15 and 16 refer to the following situation: Jared and Karla jointly bought the half meatball–half vegetarian foot-long sub shown in Fig. 3-20 for $8.00. They plan to divide the sandwich fairly using the divider-chooser method. Jared likes meatball subs three times as much as vegetarian subs; Karla is a strict vegetarian and does not eat meat at all. Assume that Jared just met Karla and has no idea that she is a vegetarian. Assume also that when the sandwich is cut, the cut is made perpendicular to the length of the sandwich. (You can describe different shares of the sandwich using the ruler and interval notation. For example, [0, 6] *describes the vegetarian half,* [6, 8] *describes one-third of the meatball half, etc.)*

$2 $6
x + 3x = 4x = $8
x = $2

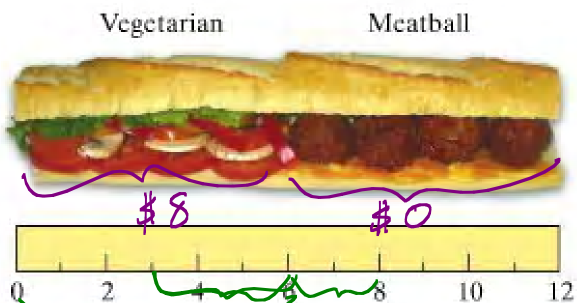


FIGURE 3-20

$2 1/2 + $6 1/3
$1 + $2 = $3

15. Suppose that they flip a coin and Jared ends up being the divider.

(a) Describe how Jared should cut the sandwich into two shares s_1 and s_2.
[0,8], s_2 = [8,12]

(b) After Jared cuts, Karla gets to choose. Specify which of the two shares Karla should choose and give the value of the share to Karla.
[0,8]
$8

16. Suppose they flip a coin and Karla ends up being the divider.

(a) Describe how Karla should cut the sandwich into two shares s_1 and s_2.
[0,3], [3,12]

(b) After Karla cuts, Jared gets to choose. Specify which of the two shares Jared should choose and give the value of the share to Jared.
[3,12]
$7
$1 of veggie (half)
$6 of meat

Ex2:

a) What is the value of meatball share to Jared? $6

b) What is the value of the meatball share to Karla? $0

c) What is the value of the [3,6] share for Karla? $4

d) What is the value of the [3,8] share for Jared? $3

Lone-Divider Method

Process:
- One person divides into n equal shares
- other choosers bid secretly on their "fair shares"
- pieces are distributed.

3 Players

Case 1 (v1): $C_1: S_1, S_3$; $C_2: S_1, S_3$

	S_1	S_2	S_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	35%	10%	55%
Cher	40%	25%	35%

Case 1 (v2): $C_1: S_2$, $C_2: S_1$, $D: S_3$

	S_1	S_2	S_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	30%	40%	30%
Cher	60%	15%	25%

Case 2:

	S_1	S_2	S_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	20%	30%	50%
Cher	10%	20%	70%

$C_1: S_3$, $C_2: S_3$ → Give $D: S_1$

Cake left

Cindy: 80%

Cher: 90%

Will get at least

Cindy: 40%

Cher: 45%

4 Players:

Case 1:

	S_1	S_2	S_3	S_4
Demi	25%	25%	25%	25%
C_1 Chan	30%	20%	35%	15%
C_2 Chloe	20%	20%	40%	20%
C_3 Chris	25%	20%	20%	35%

$C_1: S_1, \cancel{S_3}$

$C_2: S_3$

$C_3: \cancel{S_1}, S_4$

1st: $D: S_2$

next: $C_2: S_3$

next: $C_1: S_1$

next: $C_3: S_4$

Case 2:

	S_1	S_2	S_3	S_4	
Demi	25%	25%	25%	25%	
C_1 Chan	20%	20%	20%	40%	60% left
C_2 Chloe	15%	35%	30%	20%	
C_3 Chris	22%	23%	20%	35%	58% left

$C_1: S_4$

$C_2: S_2, S_3$

$C_3: S_4$

1st: $D: S_1$

$C_2: S_3$ (coin flip)

next: combine S_2 & S_4 for C_1 & C_3 to split

at least: 30% 29%

(both more than 25%)

LC: **26.** Allen, Brady, Cody, and Diane are sharing a cake valued at $20 using the lone-divider method. The divider divides the cake into four slices (s_1, s_2, s_3, and s_4). Table 3-21 shows the values of the slices in the eyes of each player.

(a) Who was the divider? Cody

(b) Find a fair division of the cake.

$5 fair share

	s_1	s_2	s_3	s_4
Allen	$4.00	$5.00	$4.00	$7.00
Brady	$6.00	$6.50	$4.00	$3.50
Cody	$5.00	$5.00	$5.00	$5.00
Diane	$7.00	$4.50	$4.00	$4.50

■ TABLE 3-21

~~s_4~~, s_4 3rd

~~s_1~~, s_2 2nd

all 4th

s_1 1st

order of actions

c) What was the actual value of the cake in the end?

$25.50

($7 + $7 + $5 + $6.5)

$\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

Lone-Chooser Method (3 players)

- 2 players do Divider-Chooser.
- each player divides their half into equal thirds.
- Chooser takes one slice from each.

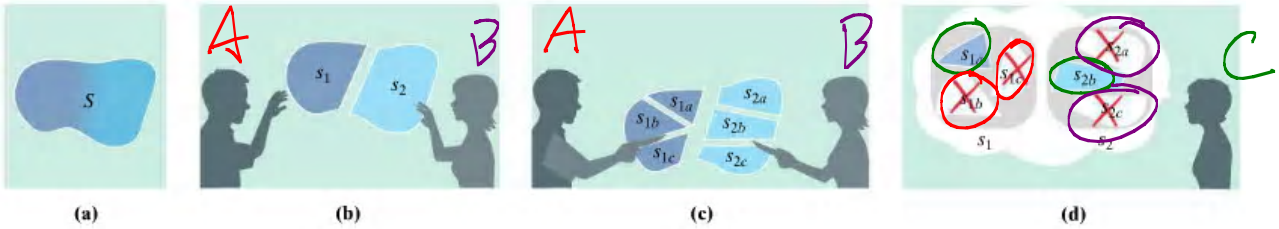


FIGURE 3-24

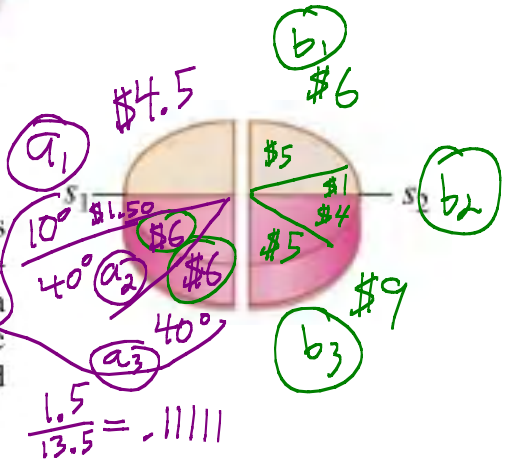
35. Suppose that Angela and Boris are the dividers and Carlos is the chooser. In the first division, Boris cuts the cake vertically through the center as shown in Fig. 3-25, with Angela choosing s_1 (the left half) and Boris s_2 (the right half). In the second division, Angela subdivides s_1 into three pieces and Boris subdivides s_2 into three pieces.

(a) Describe how Angela would subdivide s_1 into three pieces.

(b) Describe how Boris would subdivide s_2 into three pieces.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.



c) C: a_1, b_1

a: a_2, a_3

b: b_2, b_3

(c) & (d) All on LC.

LC 3(a): What is Angela's half of the cake worth? $18 (half of $36)

LC 4(a): What is one of Angela's "thirds" slice worth? $6

LC 5(b): In Boris's middle cut with both v. and s., how many degrees of vanilla are there? $\frac{1}{6} \cdot 90 = 15°$

LC 6(b): In Boris's middle cut with both v. and s., how many degrees of strawb. are there? 40°

Method of Sealed Bids – Discrete Division

- Each player bids secretly on **their** values for the items – honesty is important.
- The total value is determined for each player, as well as each players "fair share".
  - Every player could have a different fair share.
- The "Winner" of each item is given that item.
  - If they have received more than a "fair share" they must pay into the "pot" to make it fair.
- Those who did not receive a "fair share" are paid out of the pot.
  - Any extra (surplus) in the pot is divided evenly among the players.

Assumptions:

- All players must have the liquidity (cash!) to pay into the pot for items they have bid on.
- Items must all have some market value (priceless family heirlooms won't work).

44. Andre, Bea, and Chad are dividing an estate consisting of a house, a small farm, and a painting using the method of sealed bids. Table 3-25 shows the players' bids on each of the items.

	Andre	Bea	Chad
House	$150,000	$146,000	$175,000
Farm	$430,000	$425,000	$428,000
Painting	$50,000	$59,000	$57,000

630,000 630,000 660,000

■ TABLE 3-25

Fair Share: 210,000 210,000

(a) Describe the first settlement of this fair division and compute the surplus.

(b) Describe the final settlement of this fair-division problem.

Surplus: 220,000 − 151,000 − 45,000

$24,000 ÷ 3 → $8,000 more

LC 7: What is the total value to Bea?

$630,000

LC8: What is a fair share to Chad?

$220,000

LC44 a) First Settlement.

A: 210,000 − 430,000 owes ~~$220,000~~

B: 210,000 − 59,000 get: $151,000

C: 220,000 − 175,000 get $45,000

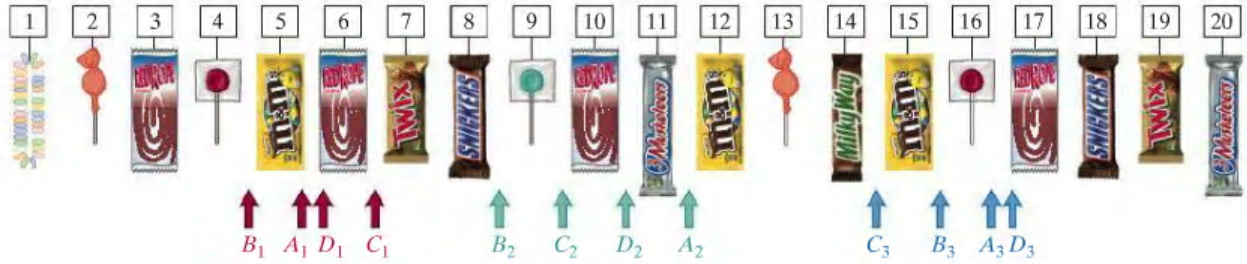
Andre: Farm, gave $212,000
Bea: Painting, got $159,000
Chad: House, got $53,000

Method of Markers

This method works well with discrete division of many, low-valued items.

Random order is established of the items

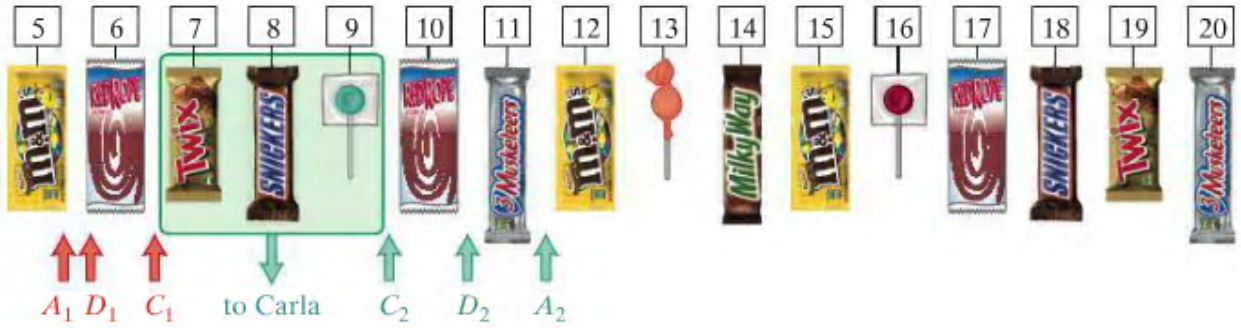
Each player will divide the set into equal shares with markers.



We go out to the first "1st-marker" and give that player their fair-share

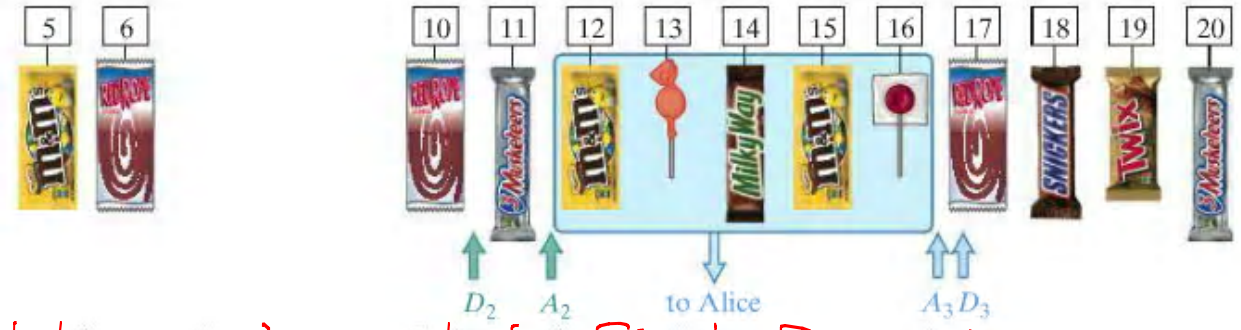
cross off all "B markers"

Find the first "2nd marker", give that player their fair share

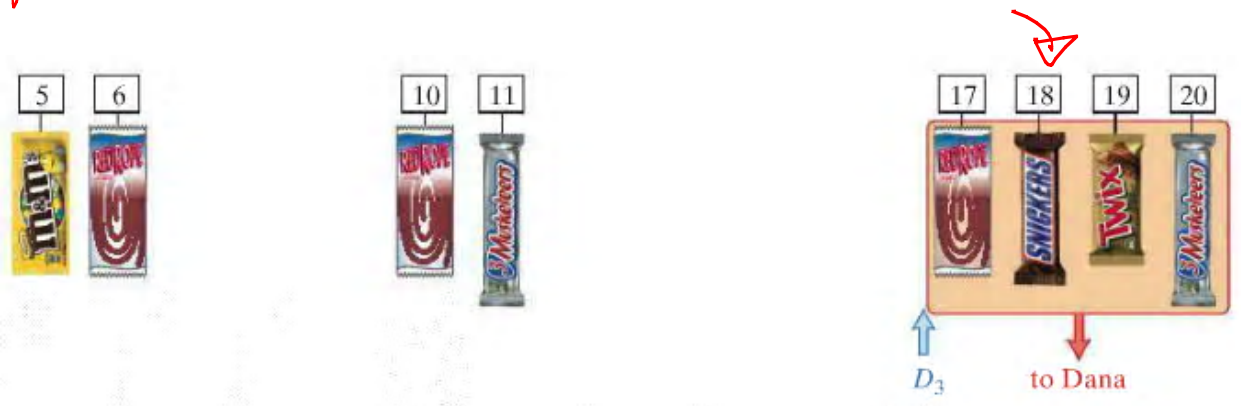


Cross off C markers

Find the first "3rd marker" - give that fair share

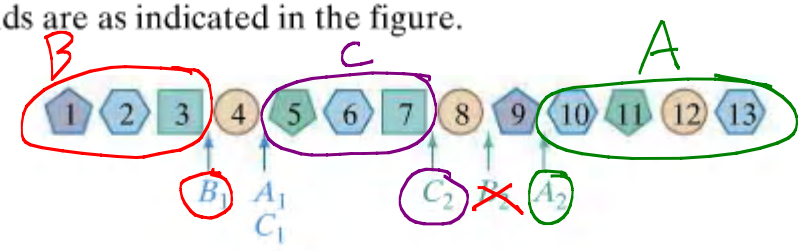


4th receives their final fair share



Disperse leftovers in some "Fair" fashion.

51. Three players (A, B, and C) are dividing the array of 13 items shown in Fig. 3-32 using the method of markers. The players' bids are as indicated in the figure.



LC a) Who gets the first group of items, and what items do they get?

B gets 1, 2, & 3

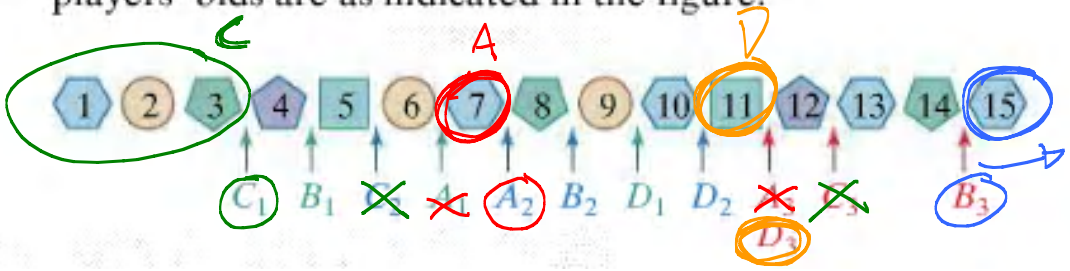
b) Who gets the second group of items, and what items do they get?

C gets 5, 6, 7

c) Who gets the third group of items, and what items do they get?

d) What items are left over? 4, 8, 9

56. Four players (A, B, C, and D) are dividing the array of 15 items shown in Fig. 3-37 using the method of markers. The players' bids are as indicated in the figure.



LC a) Who gets the first group of items, and what items do they get?

C, 1-3

b) Who gets the second group of items, and what items do they get?

A, 7

c) Who gets the third group of items, and what items do they get?

D, 11

d) Who gets the fourth group of items, and what items do they get?

B, 15

e) Left Overs? 4, 5, 6, 8, 9, 10, 12, 13, 14

50. Anne, Bess, and Cindy are roommates planning to move out of their apartment. They identify five major chores that need to be done before moving out and decide to use the method of sealed bids to reverse auction the chores. Table 3-29 shows the bids that each roommate made for each chore. Describe the final outcome of the division (which chores are done by each roommate and how much each roommate pays or gets paid.)

	Anne	Bess	Cindy
Chore 1	$20	$30	$40
Chore 2	$50	$10	$22
Chore 3	$30	$20	$15
Chore 4	$30	$20	$10
Chore 5	$20	$40	$15

LC a) What is a fair share for each?

A: $50, B: $40, C: $34

b) Which chores will Anne complete?

Chore 1

c) Which chores will Bess complete?

Chore 2

d) Which chores will Cindy complete?

Chore 3, 4, 5

$150/3 = $50 $120/3 = $40 $102/3 = $34

pay $30 pay $30 gets $6

$54 left! ÷3 ⇒ $18

e) How much does Anne Pay/Receive from the pot? (including any divided surplus)?

pays $12 ($32)

f) How much does Bess Pay/Receive from the pot? (including any divided surplus)

pays $12 ($22)

g) How much does ~~Bess~~ Cindy Pay/Receive from the pot? (including any divided surplus)

gets $24 $40 labor ⇒ ($16)