Chapter 3: Fair-Division Games

Fair Shares – 

Fair Divisions –

Necessary Assumptions:

- Rationality - Fair & rational
- Cooperation - work together
- Privacy - don’t know each others preferences
- Symmetry - everyone is even

Continuous – can be divided “infinitely” many ways (cake, land, money)

Discrete – whole items not easily broken apart (cars, paintings, houses)

Ex 1: Given the Fair-Division:

a) Who was the Divider?

   Dina

b) What are “Fair-Shares” for Angie?

   \[ s_2, s_3 \]

c) What are “Fair-Shares” for Bev?

   \[ s_1, s_2, s_4 \]

Divider-Chooser:

"You cut, I choose!"

<table>
<thead>
<tr>
<th></th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( s_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angie</td>
<td>22%</td>
<td>26%</td>
<td>28%</td>
<td>24%</td>
</tr>
<tr>
<td>Bev</td>
<td>( 25% )</td>
<td>( 26% )</td>
<td>22%</td>
<td>( 27% )</td>
</tr>
<tr>
<td>Ceci</td>
<td>20%</td>
<td>30%</td>
<td>27%</td>
<td>23%</td>
</tr>
<tr>
<td>Dina</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
</tr>
</tbody>
</table>

The divider-chooser method goes all the way back to the Old Testament: When Lot and Abraham argued over grazing rights, Abraham proposed, “Let us divide the land into left and right. If you go left, I will go right; and if you go right, I will go left” (Genesis 13:1–9).
Ex2:

a) What is the value of meatball share to Jared?

b) What is the value of the meatball share to Karla?

c) What is the value of the [3,6] share for Karla?

d) What is the value of the [3,8] share for Jared?

15. Suppose that they flip a coin and Jared ends up being the divider.

(a) Describe how Jared should cut the sandwich into two shares $s_1$ and $s_2$.

(b) After Jared cuts, Karla gets to choose. Specify which of the two shares Karla should choose and give the value of the share to Karla.

16. Suppose they flip a coin and Karla ends up being the divider.

(a) Describe how Karla should cut the sandwich into two shares $s_1$ and $s_2$.

(b) After Karla cuts, Jared gets to choose. Specify which of the two shares Jared should choose and give the value of the share to Jared.
Lone-Divider Method

Process:
- One person divides into n equal shares
- other choosers bid secretly on their "fair shares"
- pieces are distributed.

3 Players

Case 1 (v1):

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dale</td>
<td>33%</td>
<td>33%</td>
<td>33%</td>
</tr>
<tr>
<td>Cindy</td>
<td>35%</td>
<td>10%</td>
<td>55%</td>
</tr>
<tr>
<td>Cher</td>
<td>40%</td>
<td>25%</td>
<td>35%</td>
</tr>
</tbody>
</table>

Case 1 (v2):

<table>
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<th></th>
<th>$s_1$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Dale</td>
<td>33%</td>
<td>33%</td>
<td>33%</td>
</tr>
<tr>
<td>Cindy</td>
<td>30%</td>
<td>40%</td>
<td>30%</td>
</tr>
<tr>
<td>Cher</td>
<td>60%</td>
<td>15%</td>
<td>25%</td>
</tr>
</tbody>
</table>

Case 2:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dale</td>
<td>33%</td>
<td>33%</td>
<td>33%</td>
</tr>
<tr>
<td>Cindy</td>
<td>20%</td>
<td>30%</td>
<td>50%</td>
</tr>
<tr>
<td>Cher</td>
<td>10%</td>
<td>20%</td>
<td>70%</td>
</tr>
</tbody>
</table>

4 Players:

Case 1:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demi</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
</tr>
<tr>
<td>Chan</td>
<td>30%</td>
<td>20%</td>
<td>35%</td>
<td>15%</td>
</tr>
<tr>
<td>Chloe</td>
<td>20%</td>
<td>20%</td>
<td>40%</td>
<td>20%</td>
</tr>
<tr>
<td>Chris</td>
<td>25%</td>
<td>20%</td>
<td>20%</td>
<td>35%</td>
</tr>
</tbody>
</table>

4: $s_1$ leftover

Case 2:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demi</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
</tr>
<tr>
<td>Chan</td>
<td>20%</td>
<td>20%</td>
<td>40%</td>
<td>20%</td>
</tr>
<tr>
<td>Chloe</td>
<td>15%</td>
<td>35%</td>
<td>30%</td>
<td>20%</td>
</tr>
<tr>
<td>Chris</td>
<td>22%</td>
<td>23%</td>
<td>20%</td>
<td>33%</td>
</tr>
</tbody>
</table>

4: $s_1$ leftover

1. $s_1$ leftover
2. 58% left
3. $s_3$ leftover
4. 60% left

1. $s_1$ leftover
2. 58% left
3. $s_3$ leftover
4. 60% left

1st: D: $s_2$
next: C_2: $s_3$
next: C_1: $s_1$
next: C_3: $s_4$

1st: D: $s_1$
next: C_2: $s_3$
next: C_1: $s_1$
next: C_3: $s_4$
26. Allen, Brady, Cody, and Diane are sharing a cake valued at $20 using the lone-divider method. The divider divides the cake into four slices ($s_1, s_2, s_3,$ and $s_4$). Table 3-21 shows the values of the slices in the eyes of each player.

(a) Who was the divider? **Cody**

(b) Find a fair division of the cake.

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allen</td>
<td>$4.00$</td>
<td>$5.00$</td>
<td>$4.00$</td>
<td><strong>$7.00$</strong></td>
</tr>
<tr>
<td>Brady</td>
<td>$6.00$</td>
<td>$6.50$</td>
<td>$4.00$</td>
<td>$3.50$</td>
</tr>
<tr>
<td>Cody</td>
<td>$5.00$</td>
<td>$5.00$</td>
<td><strong>$5.00$</strong></td>
<td>$5.00$</td>
</tr>
<tr>
<td>Diane</td>
<td><strong>$7.00$</strong></td>
<td>$4.50$</td>
<td>$4.00$</td>
<td>$4.50$</td>
</tr>
</tbody>
</table>

**TABLE 3-21**

(c) What was the actual value of the cake in the end?

$\$25.50

($\$7 + \$7 + \$5 + \$6.5$)
Lone-Chooser Method (3 players)
- 2 players do Divider-Chooser.
- Each player divides their half into thirds.
- Chooser takes one slice from each.

LC 3(a): What is Angela’s half of the cake worth?

LC 4(a): What is one of Angela’s “thirds” slice worth?

LC 5(b): In Boris’s middle cut with both v. and s., how many degrees of vanilla are there? 

LC 6(b): In Boris’s middle cut with both v. and s., how many degrees of strawb. are there?
Method of Sealed Bids – Discrete Division

- Each player bids on their values for the items – honesty is important.
- The total value is determined for each player, as well as each players “fair share”.
  - Every player could have a different fair share.
- The “Winner” of each item is given that item.
  - If they have received more than a “fair share” they must pay into the “pot” to make it fair.
- Those who did not receive a “fair share” are paid out of the pot.
  - Any extra (surplus) in the pot is divided evenly among the players.

Assumptions:

- All players must have the liquidity (cash!) to pay into the pot for items they have bid on.
- Items must all have some market value (priceless family heirlooms won’t work).

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44. Andre, Bea, and Chad are dividing an estate consisting of a house, a small farm, and a painting using the method of sealed bids. Table 3-25 shows the players’ bids on each of the items.

<table>
<thead>
<tr>
<th></th>
<th>Andre</th>
<th>Bea</th>
<th>Chad</th>
</tr>
</thead>
<tbody>
<tr>
<td>House</td>
<td>$150,000</td>
<td>$146,000</td>
<td>$175,000</td>
</tr>
<tr>
<td>Farm</td>
<td>$430,000</td>
<td>$425,000</td>
<td>$428,000</td>
</tr>
<tr>
<td>Painting</td>
<td>$50,000</td>
<td>$59,000</td>
<td>$57,000</td>
</tr>
</tbody>
</table>

LC 7: What is the total value to Bea?
$630,000

LC 8: What is a fair share to Chad?
$220,000

LC44 a) First Settlement.

A: $210,000 - 430,000
owes $220,000.

B: $210,000 - 59,000
get: $151,000

C: $220,000 - 175,000
get $45,000

Surplus: $220,000 - 151,000 - 45,000
$24,000 ÷ 3 → $8,000 more

Andre: Farm, gave $212,000.
Bea: Painting, got $159,000.
Chad: House, got $53,000.

Method of Markers

This method works well with discrete division of many, low-valued items.

Random order is established of the items

Each player will divide the set into equal shares with markers.

We go out to the first “1st-marker” and give that player their fair-share

cross off all “B markers”
Find the first “2nd marker”, give that player their fair share.

Cross off C markers.
Find the first “3rd marker” - give that fair share.

4th receives their final fair share.

Disperse leftovers in some “fair” fashion.
51. Three players \((A, B, \text{ and } C)\) are dividing the array of 13 items shown in Fig. 3-32 using the method of markers. The players’ bids are as indicated in the figure.

\[\text{LC}\]

a) Who gets the first group of items, and what items do they get?
\[B \text{ gets } 1, 2, 3\]

b) Who gets the second group of items, and what items do they get?
\[C \text{ gets } 5, 6, 7\]

c) Who gets the third group of items, and what items do they get?

\[\text{d) What items are left over? } 4, 8, 9\]

56. Four players \((A, B, C, \text{ and } D)\) are dividing the array of 15 items shown in Fig. 3-37 using the method of markers. The players’ bids are as indicated in the figure.

\[\text{LC}\]

a) Who gets the first group of items, and what items do they get?
\[C, 1-3\]

b) Who gets the second group of items, and what items do they get?
\[A, 7\]

c) Who gets the third group of items, and what items do they get?
\[D, 11\]

d) Who gets the fourth group of items, and what items do they get?
\[B, 15\]

e) Left Overs? \[4, 5, 6, 8, 9, 10, 12, 13, 14\]
50. Anne, Bess, and Cindy are roommates planning to move out of their apartment. They identify five major chores that need to be done before moving out and decide to use the method of sealed bids to reverse auction the chores. Table 3-29 shows the bids that each roommate made for each chore. Describe the final outcome of the division (which chores are done by each roommate and how much each roommate pays or gets paid.)

<table>
<thead>
<tr>
<th></th>
<th>Anne</th>
<th>Bess</th>
<th>Cindy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chore 1</td>
<td>$20</td>
<td>$30</td>
<td>$40</td>
</tr>
<tr>
<td>Chore 2</td>
<td>$50</td>
<td>$10</td>
<td>$22</td>
</tr>
<tr>
<td>Chore 3</td>
<td>$30</td>
<td>$20</td>
<td>$15</td>
</tr>
<tr>
<td>Chore 4</td>
<td>$30</td>
<td>$20</td>
<td>$10</td>
</tr>
<tr>
<td>Chore 5</td>
<td>$20</td>
<td>$40</td>
<td>$15</td>
</tr>
</tbody>
</table>

LC a) What is a fair share for each?

A: $50, B: $40, C: $34

b) Which chores will Anne complete?

Chore 1

c) Which chores will Bess complete?

Chore 2

d) Which chores will Cindy complete?

Chore 3, 4, 5

e) How much does Anne Pay/Receive from the pot? (including any divided surplus)?

pays $12 (32)

f) How much does Bess Pay/Receive from the pot? (including any divided surplus)

pays $12 ($22)

g) How much does Cindy Pay/Receive from the pot? (including any divided surplus)

gets $24

$40 labor \rightarrow ($16)