Chapter 2: Weighted Voting   Math 107

Examples of Weighted Voting Systems

- Electoral College
- NBA Draft
- Business partnership
  40% capital, 20%, 4@ 10%

Definitions

Players – voters

Weights – how much power a player has

Quota – # of votes to pass a motion (at least a majority)

Dictator – weight $\geq$ quota

Veto Power – can prevent a motion

Dummy – voter with no power

Coalitions – set of players voting together

Grand Coalition – All players

Winning Coalition – coalition whose weight $\geq$ quota

Critical Players – required in a winning coalition

Critical Count – number of times a player is critical

quota P_1 P_2 P_3 weights

Ex 1: Looking at $\{101:99,98,3\}$, who has the power? $\{51: 49, 48, 3\}$

1/3 each! $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$, $\{P_1, P_2, P_3\}$

Ex 2 (LC): Consider the weighted voting system: [q: 6, 4, 3, 3, 2, 2].

a) What is the smallest value the quota q can take? $\boxed{11}$

b) What is the largest value the quota q can take? $\boxed{20}$

c) What is the value of q if ***at least*** three-fourths of the votes are required? $\frac{3}{4} \cdot 20 = \boxed{15}$

d) What is the value of q if ***more than*** three-fourths of the votes are required?

$\frac{3}{4}$ is $15 \rightarrow \boxed{16}$

Ex 3 (LC): In the following weighted voting systems, which players have veto power?

a) [9: 8, 4, 2, 1] P_1

b) [12: 8, 4, 2, 1] P_1, P_2

c) [14: 8, 4, 2, 1] P_1, P_2, P_3

(If they leave, it cannot pass)

Banzhaf Power Index (BPI): $\dfrac{\text{critical count each player}}{\text{total critical count}}$

COMPUTING THE BANZHAF POWER DISTRIBUTION

- **Step 1.** Make a list of all possible *winning* coalitions.
- **Step 2.** Within each winning coalition determine which are the *critical* players. (For record-keeping purposes, it is a good idea to underline each critical player.)
- **Step 3.** Find the *critical counts* $B_1, B_2, \ldots, B_N$.
- **Step 4.** Find $T = B_1 + B_2 + \cdots + B_N$.
- **Step 5.** Compute the Banzhaf power indexes: $\beta_1 = \frac{B_1}{T}, \beta_2 = \frac{B_2}{T}, \ldots, \beta_N = \frac{B_N}{T}$.

Ex 4: Find the Banzhaf power distribution of the weighted voting system.

P_1 P_2 P_3 P_4 P_5

[11: 5, 4, 3, 2, 1]

~~2 5~~ ~~3 4~~

CC: 7, 5, 3, 3, 1

Total: 19

(3) $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$

(4) $\{\underline{P_1}, \underline{P_2}, P_3, P_4\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_5\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}, P_5\}$

$\{\underline{P_1}, \underline{P_3}, \underline{P_4}, \underline{P_5}\}$ ~~P_2, P_3, P_4, P_5~~ = 10

(5) $\{\underline{P_1}, P_2, P_3, P_4, P_5\}$

B.P.D. $P_1: \frac{7}{19}$, $P_2: \frac{5}{19}$, $P_3: \frac{3}{19}$, $P_4: \frac{3}{19}$, $P_5: \frac{1}{19}$

Ex 5 (LC): Find the Banzhaf power index of Player 3. **[12: 5, 4, 3, 2, 1]**

3: $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$ 4: $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_4\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_5\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}, \underline{P_5}\}$

5: $\{\underline{P_1}, \underline{P_2}, P_3, P_4, P_5\}$

$P_3: \boxed{\frac{3}{15}} = \boxed{\frac{1}{5}}$

Shapley-Shubik Power

(Chapter 2 Continued)

Sequential coalitions – Every possible ordering of the candidates < >

Factorial - $n! = n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

4 cand.: $\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1} = 24$ coalitions $\quad 4! = 4\cdot 3\cdot 2\cdot 1 = 24$

Pivotal Player – the players whose votes turn a sequential coalition from winning to losing

Pivotal count - # of times a candidate is pivotal $\quad \langle P_1, P_3, \underline{P_4}, P_2 \rangle$

Shapley-Shubik Power Index (SSPI) – $\dfrac{\text{players pivotal count}}{\text{total \# of coallitions}}$

Ex 6 (LC): Given the following weighted voting system: **[10: 5, 4, 3, 2, 1]**

a) How many Sequential Coalitions will there be?

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = \boxed{120}$

1,2,3,4,5
1,2,3,5,4
1,2,4,3,5
1,2,4,5,3
1,2,5,3,4
1,2,5,4,3

b) Which is the pivotal player in <P_1, P_2, P_3, P_4, P_5> ?

5+4+3

c) Which is the pivotal player in <P_5, P_4, P_3, P_2, P_1> ?

1+2+3+4

COMPUTING A SHAPLEY-SHUBIK POWER DISTRIBUTION

- **Step 1.** Make a list of the $N!$ sequential coalitions with the N players.
- **Step 2.** In each sequential coalition determine *the* pivotal player. (For bookkeeping purposes underline the pivotal players.)
- **Step 3.** Find the *pivotal counts* $SS_1, SS_2, \ldots, SS_N$.
- **Step 4.** Compute the SSPIs $\sigma_1 = \frac{SS_1}{N!}, \sigma_2 = \frac{SS_2}{N!}, \ldots, \sigma_N = \frac{SS_N}{N!}$.

$3! = 3 \cdot 2 \cdot 1 = 6$

Ex 7: Find the Shapley-Shubik Power Distribution of [16: 9, 8, 7]

$\langle P_1, \underset{9+8}{P_2}, P_3 \rangle$ $\langle P_2, \underset{8+9}{P_1}, P_3 \rangle$ $\langle P_3, \underset{7+9}{P_1}, P_2 \rangle$

$\langle P_1, \underset{9+7}{P_3}, P_2 \rangle$ $\langle P_2, P_3, \underset{8+7}{P_1} \rangle$ $\langle P_3, P_2, P_1 \rangle$

$\sigma_1 = \frac{4}{6}$ $\sigma_2 = \frac{1}{6}$ $\sigma_3 = \frac{1}{6}$

Ex 8: List all of the Sequential Coalitions of [q: P_1, P_2, P_3, P_4,]. *(if time permits)*

Cheat

$\langle P_1, P_2, P_3, P_4 \rangle$ $\langle P_2, P_1, P_3, P_4 \rangle$ $\langle 3, 1, 2, 4 \rangle$ $\langle 4, 1, 2, 3 \rangle$

$\langle P_1, P_2, P_4, P_3 \rangle$ $\langle P_2, P_1, P_4, P_3 \rangle$ $\langle 3, 1, 4, 2 \rangle$ $\langle 4, 1, 3, 2 \rangle$

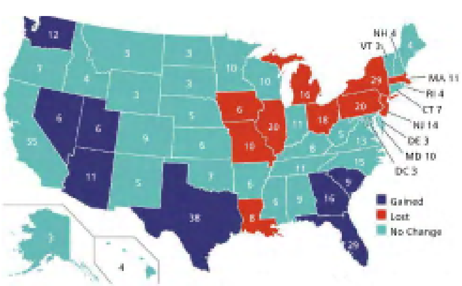
$\langle P_1, P_3, P_2, P_4 \rangle$ $\langle P_2, P_3, P_1, P_4 \rangle$ $\langle 3, 2, 1, 4 \rangle$ $\langle 4, 2, 1, 3 \rangle$

$\langle P_1, P_3, P_4, P_2 \rangle$ $\langle P_2, P_3, P_4, P_1 \rangle$ $\langle 3, 2, 4, 1 \rangle$ $\langle 4, 2, 3, 1 \rangle$

$\langle P_1, P_4, P_2, P_3 \rangle$ $\langle P_2, P_4, P_1, P_3 \rangle$ $\langle 3, 4, 1, 2 \rangle$ $\langle 4, 3, 1, 2 \rangle$

$\langle P_1, P_4, P_3, P_2 \rangle$ $\langle P_2, P_4, P_3, P_1 \rangle$ $\langle 3, 4, 2, 1 \rangle$ $\langle 4, 3, 2, 1 \rangle$

The Electoral College



State	Weight	Shapley-Shubik	Banzhaf
Alabama	9	1.64%	1.64%
Alaska	3	0.54%	0.55%
Arizona	11	2.0%	2.0%
Arkansas	6	1.1%	1.1%
California	55	11.0%	11.4%
Colorado	9	1.64%	1.64%
Connecticut	7	1.3%	1.3%
Delaware	3	0.54%	0.55%
District of Columbia	3	0.54%	0.55%
Florida	29	5.5%	5.4%
Georgia	16	2.9%	2.9%
Hawaii	4	0.72%	0.73%
Idaho	4	0.72%	0.73%
Illinois	20	3.7%	3.7%
Indiana	11	2.0%	2.0%
Iowa	6	1.1%	1.1%
Kansas	6	1.1%	1.1%
Kentucky	8	1.5%	1.5%
Louisiana	8	1.5%	1.5%
Maine*	4	0.72%	0.73%
Maryland	10	1.8%	1.8%
Massachusetts	11	2.0%	2.0%
Michigan	16	2.9%	2.9%
Minnesota	10	1.8%	1.8%
Mississippi	6	1.1%	1.1%
Missouri	10	1.8%	1.8%

State	Weight	Shapley-Shubik	Banzhaf
Montana	3	0.54%	0.55%
Nebraska*	5	0.9%	0.91%
Nevada	6	1.1%	1.1%
New Hampshire	4	0.72%	0.73%
New Jersey	14	2.6%	2.6%
New Mexico	5	0.9%	0.91%
New York	29	5.5%	5.4%
North Carolina	15	2.8%	2.7%
North Dakota	3	0.54%	0.55%
Ohio	18	3.3%	3.3%
Oklahoma	7	1.3%	1.3%
Oregon	7	1.3%	1.3%
Pennsylvania	20	3.7%	3.7%
Rhode Island	4	0.72%	0.73%
South Carolina	9	1.64%	1.64%
South Dakota	3	0.54%	0.55%
Tennessee	11	2.0%	2.0%
Texas	38	7.3%	7.2%
Utah	6	1.1%	1.1%
Vermont	3	0.54%	0.55%
Virginia	13	2.4%	2.4%
Washington	12	2.2%	2.2%
West Virginia	5	0.9%	0.91%
Wisconsin	10	1.8%	1.8%
Wyoming	3	0.54%	0.55%