Chapter 16: Probabilities, Odds, and Expectations

Math 107

Random Experiment: unknown outcome tossing a coin, rolling dice, drawing cards, free throws, etc

Sample Space: set of all possible outcomes

Ex 1) Tossing a coin

$S = \{$ Head, Tail $\}$ $\qquad S = \{H, T\}$

$N =$ 2

Ex 2) Tossing a coin twice

$S = \{HH, HT, TH, TT\}$

$N =$ 4

Number of heads?

$S = \{0, 1, 2\}$

$N =$ 3

Ex 3) The sum of rolling a pair of dice.

$\boxed{\cdot}\ \boxed{:}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}$

smallest: 2 $\boxed{\cdot}\boxed{\cdot}$

largest: 12 $\boxed{6}\boxed{6}$

Most common: 7 (many ways)

$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$N =$ 11

Deck of Cards: Four suits: clubs, diamonds, hearts, spades

13 cards in each suit

2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A (2 jokers)

Ex 4) a) Drawing a card of a certain suit from a 52-card deck.

$S = \{c, d, h, s\}$

$N = 4$

b) Drawing a card of a certain value from a 52-card deck.

$S = \{2, 3, 4, \ldots, 10, J, Q, K, A\}$

$N = 13$

An Event is a subset of the sample space. Any subset can be considered.

A set with N elements has 2^N subsets, so there are 2^N Events.

Ex 5) Tossing a coin twice $S = \{HH, HT, TH, TT\}$, $N = 4$ $2^4 = 16$ events

1. $\{\ \}$	5. $\{TT\}$	9. $\{HT, TH\}$	13. $\{HH, HT, TT\}$
2. $\{HH\}$	6. $\{HH, HT\}$	10. $\{HT, TT\}$	14. $\{HH, TH, TT\}$
3. $\{HT\}$	7. $\{HH, TH\}$	11. $\{TH, TT\}$	15. $\{HT, TH, TT\}$
4. $\{TH\}$	8. $\{HH, TT\}$	12. $\{HH, HT, TH\}$	16. $\{HH, HT, TH, TT\}$

Impossible Event: empty set

Certain Event: whole sample space

Simple Event: one outcome (2, 3, 4, 5)

(LC) Which event is: "2nd toss a head"? $\{HH, TH\}$

(LC) Which event is: "At least one tail"? $\{HT, TH, TT\}$

~~HH~~, HT, TH, TT

$\underline{2} \cdot \underline{2} \cdot \underline{2} = 8$

Ex 5) Tossing a Coin 3 Times

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT \}$

$N = 8$

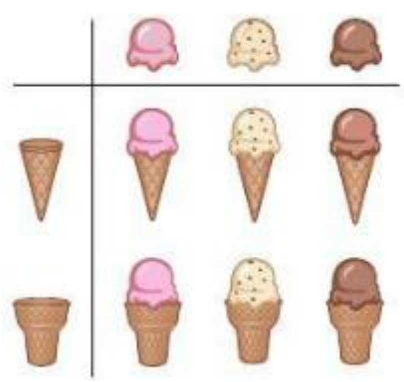
(LC) What is the total number of possible events? $2^8 = 256$ events

Event (plain English description)	Event (set description)
All tosses come up heads	$\{HHH\}$
All tosses come up the same	$\{HHH, TTT\}$
Toss one head and two tails	$\{HTT, THT, TTH\}$
First toss comes up heads	$\{HTT, HTH, HHT, HHH\}$
The tosses don't come up all the same	$\{HHT, HTH, HTT, THH, THT, TTH\}$
Toss more heads than tails	$\{HHT, HTH, THH, HHH\}$
Toss an equal number of heads and tails	$\{\ \}$ (the *impossible* event)
Toss three or fewer heads	S (the certain event)

$\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$

Multiplication Rule: If there are *m* ways to do *X*, and *n* ways to do *Y*, then *X* and *Y* together (and in that order) can be done in $m \cdot n$ ways.

Ex 6) Ice Cream Cones –



a) If you have 2 types of cones, and 3 types of ice cream, how many different choices are there?

$2 \cdot 3 = \boxed{6 \text{ choices}}$

b) **(LC)** If What's the Scoop? ice cream parlor has 3 cone choices, 20 flavors of ice cream, and 7 toppings to choose from, how many possibilities are there if you use one cone, one flavor, and one topping?

$3 \cdot 20 \cdot 7 = 420$

c) **(LC)** Now add the possibility of a bowl and no topping, how many possibilities?

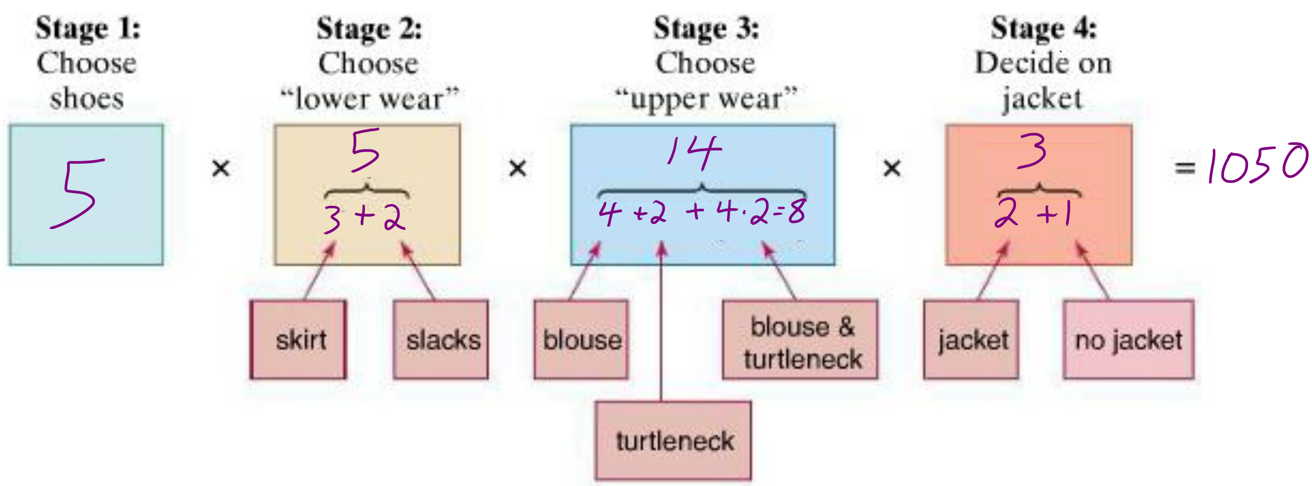
$4 \cdot 20 \cdot 8 = 640$

Ex 7) Packing for a business Trip – all matching clothes

__5__ shoes __3__ skirts __4__ blouses __2__ jackets

$5 \cdot 3 \cdot 4 \cdot 2 = 120$ outfits

_5_ shoes _3_ skirts _2_ slacks _4_ blouses _2_ turtlenecks _2_ jackets



Ex 8) More Ice Cream: True Double – Suppose an ice cream parlor has 20 ice cream flavors. A True Double is two scoops that are not the same. How many options are there?

$\underline{20} \times \underline{19} = 380$ is double!

because C & V = V & C

$\dfrac{380}{2} = \boxed{190}$

$20 \cdot 19 \cdot 18 = \dfrac{20!}{3!17!}$

Ex 9) True Triple $\underline{20} \times \underline{19} \times \underline{18} = 6840$

{CSV, CVS, SVC, SCV, VSC, VCS}

$\underline{3} \cdot \underline{2} \cdot \underline{1} = 6$ similar combos

$\dfrac{6840}{6} = \boxed{1140}$

Ex 10) Number of 5 Card Stud Hands vs. 5-Card Draw Hands

$\dfrac{52!}{47!}$ $\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48} = 311{,}875{,}200$ (order matters!)

$\dfrac{52!}{5!47!}$ $\dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120} = 2{,}598{,}960$ (order doesn't matter!)

"n things taken r at a time."

Permutations: order matters! $\boxed{\text{Permutation locks!}}$ "n choose r"

$${}_nP_r = \frac{n!}{(n-r)!}$$

Combinations: order doesn't matter! ~~Not combination locks!~~

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$

Ex 11) With 20 flavors of ice cream:

a) How many True Quadruples (4 flavors) can you have in a bowl?

Permutation or Combination ?

order doesn't matter

$${}_{20}C_4 = \frac{20!}{4!\,16!} = \frac{\overset{5}{\cancel{20}} \cdot 19 \cdot \overset{3}{\cancel{18}} \cdot 17 \cdot \cancel{16!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{16!}} = \boxed{4845}$$

b) **(LC)** How many True Quadruples (4 flavors) can you have on a cone, where you care about what order you will each them in?

Permutation or Combination ?

$\underline{20} \cdot \underline{19} \cdot \underline{18} \cdot \underline{17}$

$${}_{20}P_4 = \frac{20!}{(20-4)!} = \frac{20!}{16!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot \cancel{16!}}{\cancel{16!}} = \boxed{116{,}280}$$

Probabilites: The chance that an event will happen, between a scale of 0 (impossible event) to 1 (whole sample space = certain event).

We use Pr(E)= and the sum of all the probabilities of a Sample Space must add up to 1.

$$\underbrace{Pr(HH) = .25 \qquad Pr(HT, TH) = .5 \qquad Pr(TT) = .25}_{1}$$

Ex 12) If we know the probability of 4 out of 5 golfers to win a women's golf tournament, the 5th wil be...

$1 - (.2 + .16 + .25 + .12) = 1 - .73 = .27$

Pr(A)=0.2 Pr(B)=0.16 Pr(C)=0.25 Pr(D)=0.12, then Pr(E)=_.27_____

What is the probability a man will win the tournament?

Pr(Man winning) = 0

- **Sample space:** $S = \{o_1, o_2, \ldots, o_N\}$.
- **Probability assignment:** $\Pr(o_1), \Pr(o_2), \ldots, \Pr(o_N)$.
  [Each of these is a number between 0 and 1 satisfying $\Pr(o_1) + \Pr(o_2) + \cdots + \Pr(o_N) = 1$.]
- **Events:** These are all the subsets of S, including $\{\ \}$ and S itself. The probability of an event is given by the sum of the probabilities of the individual outcomes that make up the event. [In particular, $\Pr(\{\ \}) = 0$ and $\Pr(S) = 1$.]

- **Probability of an event:** If k denotes the size of an event E and N denotes the size of the sample space S, then in an equiprobable space $\Pr(E) = k/N$.

Ex 13) Rolling a pair of Dice: Record the number of times you roll each

	~~Your Recorded rolls~~ Class	Total	Prob	Class Total	Class Prob
Roll a 2	0+1+1+1+0+1+2+1+0+2	9		9	9/360
Roll a 3	3+1+6+5+1+1+1+3+3+0	24		24	24/360
Roll a 4	2+2+5+2+1+0+1+3+3+2	21		21	21/360
Roll a 5	5+4+3+5+4+4+3+4+3+5	40		40	40/360
Roll a 6	5+4+2+5+6+6+7+5+8+7	55		55	55/360
Roll a 7	10+7+6+2+0+4+6+4+5+7	51		51	51/360
Roll a 8	2+8+6+5+8+5+1+5+3+2	45		45	45/360
Roll a 9	2+8+4+3+7+7+6+4+3+5	49		49	49/360
Roll a 10	4+0+0+6+4+4+2+5+4+2	31		31	31/360
Roll a 11	1+0+2+2+3+4+5+2+4+2	25		25	25/360
Roll a 12	2+1+1+0+2+0+2+0+0+2	10		10	10/360
	Total Rolls	360		360	

Actual Probability

$\underline{6} \cdot \underline{6} = 36$

	k	Pr		k	Pr
Roll a 2 11	1	1/36	Roll a 8	5	5/36
Roll a 3 12, 21	2	2/36	Roll a 9	4	4/36
Roll a 4 13, 31, 22	3	3/36	Roll a 10	3	3/36
Roll a 5 14, 23, 32, 41	4	4/36	Roll a 11	2	2/36
Roll a 6 15, 24, 33, 42, 51	5	5/36	Roll a 12	1	1/36
Roll a 7 16, 25, 34, 43, 52, 61	6	6/36			

Ex 14) Rolling two dice: Pr($\overbrace{\text{at least one "boxcar" is rolled}}^{B}$) – boxcar = 6

Tallying: 16, 26, 36, 46, 56, 66, 65, 64, 63, 62, 61

$Pr(B) = 11/36$

Complimentary Events: the opposite of an event and itself

$\boxed{Pr + Pr = 1}$

$\underline{5} \cdot \underline{5} = 25 \quad Pr(\sim B) = \frac{25}{36} \quad Pr(B) = 1 - \frac{25}{36} = \frac{11}{36}$

Independent Events:

Pr(1st dice not "boxcar")= $5/6$

Pr(2nd dice not "boxcar")= $5/6$

$Pr(Both) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36}$

$Pr(B) = \boxed{\frac{11}{36}}$

Multiplication principle for Independent Events: Pr(*E* and *F*) = Pr(*E*)Pr(*F*)

Ex 15) Rolling an honest die 4 times – If at least one roll is a boxcar, you win. Find the Probability of winning.

$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{1296}$

opposite: every roll is not a boxcar

$Pr(\sim \text{boxcar}) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{625}{1296} = .4823$

$1 - .4823$

$\boxed{Pr(\text{at least 1 boxcar}) = .5177}$

Ex 16) What is the probability of "4 of a kind" in a 5-card draw hand of poker?

$_{52}C_5 = \frac{\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48} \cdot 47!}{5!\ 47!} = \frac{311,875,200}{120} = 2,598,960$

$\underbrace{A, A, A, A}_{1\text{ way}} \cdot \underline{48} = 48$ ways to have 4 aces

13 values w/ 48 combos each

$13 \cdot 48 = 624$ possible 4 of a kind hands

$\frac{624}{2,598,960} \approx .00024 = .02\%$ or $\frac{1}{4165}$

Odds: Let E be an arbitrary event.

- F is the number of ways E can occur (favorable)
- U is the number of ways E cannot occur (unfavorable)

Odds of (or odds in favor of) event E are the ***ratio*** F to U.

Odds against event E are the ***ratio*** U to F.

Ex 17) Find the odds of rolling a "natural" (7 or 11) with two dice. $N=36$

# of ways: 6 + 2 = 8 favorable

28 unfavorable ← (36−8)

Odds are 2 to 7 $\quad (\underset{f}{8} \text{ to } \underset{u}{28}) \div 4$

Ex 18) Find the odds of each of the following events

a) An event E with Pr(E)=3/11.

$\boxed{\text{Odds are } 3 \text{ to } 8}$

b) An event F with Pr(F)=0.375

odds are .375 to .625

$\frac{3}{8}$ to $\frac{5}{8}$ ⇒ $\boxed{3 \text{ to } 5}$

Ex 19) Three candidates – Aguilera, Bieber, and Cyrus – are running for mayor of Cleansburg. The odds of Aquilera winning are 1 to 2, and the odds of Cyrus winning are 2 to 7. What are the odds of Bieber winning?

$Pr(\text{odds}) = \frac{F}{F+U}$

A: $\frac{1}{3} = \frac{3}{9}$

C: $\frac{2}{9}$

B: $4/9$

Odds of Justin: 4 to 5
(Bieber)

Expectations

Ex 20) The current "weights" of our class are as follows:

Homework: 12%

Quizzes: 13%

Exams: 50%

Group Projects: 25%

If your scores for each category are 85% HW, 75% Quizzes, 83% Exams, 90% GP, what is your overall grade in the class?

$.85(.12)+.75(.13)+.83(.50)+.9(.25)=83.95\%$

Ex 21) At Thomas Jefferson High School, the student body is divided by age as follows: 7% of the students are 14, 22% of the students are 15, 24% of the students are 16, 23% of the students are 17, 19% of the students are 18, and the rest of the students are 19. Find the average age of the students at Thomas Jefferson High School. 5%

$.07(14)+.22(15)+.24(16)+.23(17)+.19(18)+.05(19)=\boxed{16.4}$

- **Weighted average:** Let X be a variable that takes the values $x_1, x_2, \ldots, x_N$, and let $w_1, w_2, \ldots, w_N$ denote the respective weights for these values, with $w_1 + w_2 + \cdots + w_N = 1$. The **weighted average** for X is given by

$$w_1 \cdot x_1 + w_2 \cdot x_2 + \cdots + w_N \cdot x_N$$

Ex 22) Guessing on the SAT

Option	Leave blank	A	B	C	D	E
Payoff	0	−0.25	1	−0.25	−0.25	−0.25

Outcome	Correct answer (B)	Incorrect answer (A, C, D, or E)
Point payoff	1	−0.25
Probability	0.2	0.8

$0.2(1)+0.8(-0.25)=0.2-0.2$

$=0$

Ex 23) Guessing at the SAT – Part 2.

What if you can definitely rule out one of the choices?

Outcome	Correct answer (B)	Incorrect answer (A, C, or D)
Point payoff	1	−0.25
Probability	0.25	0.75

$.25(1) + .75(-.25) = .0625$ points

What if you can definitely rule out two of the choices?

Outcome	Correct answer (B)	Incorrect answer (A or C)
Point payoff	1	$-1/4$
Probability	$1/3$	$2/3$

$\frac{1}{3}(1) + \frac{2}{3}\left(-\frac{1}{4}\right) = \frac{1}{6}$ point

- **Expectation:** Suppose X is a variable that takes on numerical values $x_1, x_2, x_3, \ldots, x_N$, with probabilities $p_1, p_2, p_3, \ldots, p_N$, respectively. The *expectation* (or *expected value*) of X is given by

$$E = p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 + \cdots + p_N \cdot x_N$$

Ex 24) A fair coin is tossed three times. Find the expected number of heads that come up. Statistically, what would your guess be?

$\{HHH, \underbrace{2H1T}_{3}, \underbrace{1H2T}_{3}, TTT\}$

$Pr(3H) = 1/8$

$Pr(2H) = 3/8$, $Pr(1H) = 3/8$, $Pr(0H) = 1/8$

$\frac{1}{8} \cdot 3 + \frac{3}{8} \cdot 2 + \frac{3}{8} \cdot 1 + \frac{1}{8} \cdot 0$

$= 1.5$

Ex 25) A basketball player shoots two consecutive free throws. Each free-throw is worth 1 point and has probability of success $p = 3/4$. Let X denote the number of points scored. Find the expected value of X.

$\{ss, sf, fs, ff\}$

$Pr(ss) = Pr(s) \cdot P(s) = 3/4 \cdot 3/4 = 9/16$

$Pr(sf) = Pr(s) \cdot P(f) = 3/4 \cdot 1/4 = 3/16$

$Pr(fs) = \quad = \quad = 3/16$

$Pr(ff) = Pr(f) Pr(f) = 1/4 \cdot 1/4 = 1/16$

$\left(\frac{9}{16}\right) 2\text{pts} + \left(\frac{6}{16}\right) 1\text{pt} + \left(\frac{1}{16}\right) 0\text{pts}$

$\frac{18}{16} + \frac{6}{16} = \frac{24}{16} = \boxed{1.5 \text{ pts}}$