6.3 – Orthogonal Projections
6.4 – Gram-Schmidt Process

Math 220

Warnock - Class Notes

Given a vector $\mathbf{y}$ and a subspace W in $\mathbb{R}^n$ there is a vector $\hat{\mathbf{y}} \in W$ such that

1) $\hat{\mathbf{y}}$ is the unique vector in *W* for which $\underbrace{\mathbf{y}-\hat{\mathbf{y}}}_{z}$ is orthogonal to *W*

2) $\hat{\mathbf{y}}$ is the unique vector in *W* closest to $\mathbf{y}$

Theorem 8 The Orthogonal Decomposition Theorem

Let *W* be a subspace of $\mathbb{R}^n$. Then each **y** in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \qquad (1)$$

where $\hat{\mathbf{y}}$ is in *W* and **z** is in $W^\perp$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of *W*, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \qquad (2)$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Ex 1: Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\mathbf{y} = \begin{bmatrix}1\\3\\5\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}$$

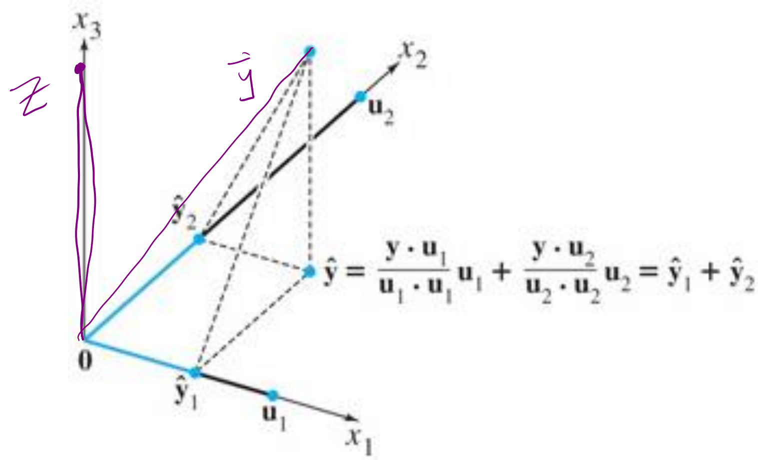
orthogonal ✓

$y = \hat{y} + z$

$$\hat{y} = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2 = \frac{\overbrace{1+9-10}^{0}}{1+9+4}\begin{bmatrix}1\\3\\-2\end{bmatrix} + \frac{5+3+20}{25+1+16}\begin{bmatrix}5\\1\\4\end{bmatrix} = \frac{28}{42}\begin{bmatrix}5\\1\\4\end{bmatrix} = \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix}1\\3\\5\end{bmatrix} - \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix} = \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}$$

$$y = \overbrace{\begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix}}^{\hat{y}} + \overbrace{\begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}}^{z}$$



Theorem 9 The Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\widehat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\widehat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \widehat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \qquad (3)$$

for all $\mathbf{v}$ in W distinct from $\widehat{\mathbf{y}}$.

Ex 2: As in Ex 1, $\begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$ is the closest point in $W = \text{Span}\left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} \right\}$ to $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Find the distance from $\mathbf{y}$ to W

$\|y - \hat{y}\| = \|z\| = \sqrt{49/9 + 49/9 + 49/9} = \boxed{\frac{7}{3}\sqrt{3}}$

$\begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$ $\quad \frac{49}{9}\cdot 3$

Practice Problems

1. Let $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$, and $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$.

Use the fact that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to compute $\text{proj}_W\, \mathbf{y}$. $= \hat{y}$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{63+1+24}{49+1+16} u_1 + \frac{9+1-12}{1+1+4} \vec{u}_2 = \frac{88}{66} \vec{u}_1 + \frac{-2}{6} \vec{u}_2$$

$$= \frac{4}{3} \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -28/3 + 1/3 \\ 4/3 - 1/3 \\ 16/3 + 2/3 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix} = y$$

$\Longrightarrow y \in W$

$z = \vec{0}$

2. Let *W* be the subspace spanned by the **u**'s, and write **y** as the sum of a vector in *W* and a vector orthogonal to *W*.

$$\mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$y = \hat{y} + z$, $\hat{y} \in W$, $z \in W^\perp$

$(\quad \hat{y} \quad) + (\quad z \quad)$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3 = \frac{6}{3} u_1 + \frac{10}{15} u_2 + \frac{-2}{3} u_3$$

$$= \begin{bmatrix} 2 - 2/3 + 2/3 \\ 2 + 2 - 0 \\ 0 + 2/3 - 2/3 \\ 2 - 4/3 - 2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$z = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

The Gram-Schmidt Process

Ex 3: Let $W = \text{Span}\left\{ \mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \right\}$, construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$v_1 = x_1$ find a proj of x_2 onto x_1, then $\vec{v}_2 = x_2 - \hat{x}_2$

$$(z=)\ v_2 = x_2 - \frac{x_2 \cdot x_1}{x_1 \cdot x_1} x_1 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \frac{24+0+6}{9+0+1} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}$$

Ex 4:

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus is a basis for a subspace W of $\mathbb{R}^4$. Construct an orthogonal basis for W.

$v_1 = x_1$

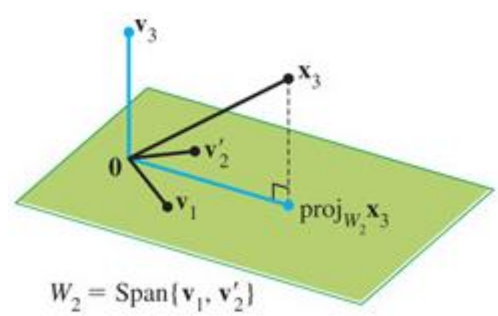
$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$v_2' = 4 v_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2'}{v_2' \cdot v_2'} v_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 - 1/2 + 1/2 \\ 0 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(the second vector marked "or v_2", the third marked "or v_3")



Theorem 11 The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a nonzero subspace *W* of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\ \vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for *W*. In addition

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p$$

The result of this is that every nonzero subspace W in $\mathbb{R}^n$ has an orthogonal basis.

An orthonormal basis is constructed easily by normalizing all the $\mathbf{v}_k$'s to unit vectors.

Ex 5: Re-write the orthogonal basis found in Ex 4 as an orthonormal basis.

$$\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

$\|v_1\| = \sqrt{4} = 2$

$\|v_2\| = \sqrt{12} = 2\sqrt{3}$

$\|v_3\| = \sqrt{6}$

$$\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -3/2\sqrt{3} \\ 1/2\sqrt{3} \\ 1/2\sqrt{3} \\ 1/2\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

Practice Problems

1. Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

Construct an orthonormal basis for *W*.

$V_1 = X_1$

$$V_2 = X_2 - \frac{X_2 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} - \frac{1/3 + 1/3 - 2/3}{3} V_1 = V_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$\|X_1\| = \sqrt{3}$

$\|X_2\| = \sqrt{6/9} = \sqrt{6}/3$

$$\frac{X_1}{\|X_1\|} \rightarrow \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \right\} \leftarrow \frac{X_2}{\|X_2\|}$$

2. Use the Gram–Schmidt process to produce an orthogonal basis for *W*.

$$W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \text{ where } \mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

$V_1 = X_1$

$$V_2 = X_2 - \frac{X_2 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 + 3(-1) \\ -8 + 3(3) \\ -2 + 3(1) \\ -4 + 3(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$V_3 = X_3 - \frac{X_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{X_3 \cdot V_2}{V_2 \cdot V_2} V_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \left(\tfrac{5}{2}\right) \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 + 1/2 - 15/2 \\ 3 - 3/2 - 5/2 \\ 6 - 1/2 - 5/2 \\ -3 - 1/2 + 5/2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$$