<u>6.2 – Orthogonal Sets</u>

-

Warnock - Class Notes

A set of vectors $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ is called an <u>Orthogonal</u> <u>Set</u> if each pair of distinct vectors from the set is orthogonal. That is, $\underline{u_i \cdot u_j} = O$ when $i \neq j$.

Ex 1: Determine whether the set of vectors is orthogonal.

a)
$$\begin{bmatrix} u_{1} & u_{2} & u_{3} \\ -6 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$
 $u_{1} \cdot u_{2} = 6 - 7 + 1 = 0$ $v = 5$ $u_{1} = 5$ $v = 5$ v

b)
$$\begin{bmatrix} 3\\ -2\\ 1\\ 3\\ \end{bmatrix}, \begin{bmatrix} -1\\ 3\\ -3\\ 4\\ \end{bmatrix}, \begin{bmatrix} 3\\ 8\\ 7\\ 0\\ \end{bmatrix}$$
 $U_1 \cdot U_2 = -3 - 6 - 3 + 12 = 0 \vee$ Orthogonal Set

Theorem 4

If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof:
$$\vec{O} = C_1 \vec{u}_1 + C_2 \vec{u}_2 + \dots + C_p \vec{u}_p$$
 (Show $C_i = 0, i \le i \le p$)
 $\vec{O} \cdot \vec{u}_1 = (C_1 \vec{u}_1 + C_2 \vec{u}_2 + \dots + C_p \vec{u}_p) \cdot \vec{u}_1$
 $\vec{O} = C_1 \vec{u}_1 \cdot \vec{u}_1 + C_2 \vec{u}_2 \cdot \vec{u}_1 + \dots + C_p \vec{u}_p \cdot \vec{u}_1$
 $\vec{U}_1 \cdot \vec{u}_1 = 0, i \ne 1$
 $\vec{O} = C_1 \vec{u}_1 \cdot \vec{u}_1$ ($\vec{u}_1 \cdot \vec{u}_1 \ne 0, \text{ since } \vec{u}_1 \text{ is non-zero vector}$)
 $\Rightarrow C_1 = 0$
Similarly, $C_2 = C_3 = \dots = C_p$ are zero \therefore Siz linearly independent

Definition

An **orthogonal basis** for a subspace W of \mathbb{R}^n is a basis for W that is also an orthogonal set.

Theorem 5

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of \mathbb{R}^n . For each **y** in W, the weights in the linear combination

are given by

(_______

$$c_j = rac{{f y} \cdot {f u}_j}{{f u}_j \cdot {f u}_j} \hspace{1cm} (j=1,\ldots,p)$$

Ex 2: The vector $\mathbf{v} = \begin{bmatrix} 4 \\ -8 \\ -10 \\ 17 \end{bmatrix}$ is in the subspace *W* with orthogonal basis from Ex 1b). $\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Express **v** as a linear combination of the orthogonal basis.

$$= \frac{\vec{v} \cdot \vec{u}_{i}}{\vec{u}_{i} \cdot \vec{u}_{i}} = \frac{12 + 16 - 10 + 51}{9 + 4 + 1 + 9} = \frac{69}{23} = 3$$

$$\begin{bmatrix} 3\\-2\\1\\3\end{bmatrix}, \begin{bmatrix} -1\\3\\-3\\4\end{bmatrix}, \begin{bmatrix} 3\\8\\7\\0\end{bmatrix}$$

 $\vec{V} = 3\vec{u}_1 + 2\vec{u}_2 - \vec{u}_3$

U;u;= O Viti

$$C_{2} = \frac{\vec{V} \cdot \vec{U}_{2}}{\vec{u}_{a} \cdot \vec{u}_{a}} = \frac{-4 - 24 + 30 + 68}{1 + 9 + 9 + 16} = \frac{70}{35} = 2$$

$$C_{3} = \frac{\vec{V} \cdot \vec{u}_{3}}{\vec{u}_{3} \cdot \vec{u}_{3}} = \frac{12 - 64 - 70 + 0}{9 + 64 + 49 + 0} = \frac{-122}{122} = -1$$

An Orthogonal Projection
$$y = \frac{4}{3} + z$$

 $y = \propto u$
 $Q = Z \cdot u = (y - \alpha u) \cdot u$
 $Q = y \cdot u - \alpha u \cdot u$
 $Q = y \cdot u - \alpha u \cdot u$
 $Q = y \cdot u - \alpha u \cdot u$
 $Q = \frac{y \cdot u}{u \cdot u} = D \left[\underbrace{y} = \frac{y \cdot u}{u \cdot u} \underbrace{y} \right] \text{ orthogonal projection of y onto u}$
 $y = \frac{y}{2} + \frac{z}{u \cdot u} \underbrace{\text{component of y out}}_{Q = \frac{y}{2} + \frac{z}{u}}_{U = \frac{y}{u} = \frac{y}{u}} \mathbf{u}$
 $\mathbf{y} = \mathbf{y} + \frac{z}{u} \underbrace{\text{component of y out}}_{Q = \frac{y}{2} + \frac{z}{u}}_{U = \frac{y}{u} = \frac{y}{u}} \mathbf{u}$
 $\mathbf{x} = \frac{y \cdot u}{u \cdot u} = D \underbrace{y}_{u \cdot u} = \frac{y}{2} + \frac{z}{u} \underbrace{y}_{u \cdot u} \mathbf{u}$
 $\mathbf{x} = \frac{y \cdot y}{u \cdot u} = \frac{y}{u} \underbrace{y}_{u \cdot u} \mathbf{u}$
Ex 3: Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.
Then write $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ as a sum of two orthogonal vectors. Also, observe geometrically.
 $4 = \underbrace{y}_{u \cdot x} \underbrace{u}_{u \cdot x} = \underbrace{-\frac{4+1}{4}}_{U \cdot t} \begin{bmatrix} -\frac{4}{2} \end{bmatrix} = \underbrace{-\frac{1}{2}}_{U \cdot x} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$
 $\mathbf{y} = \underbrace{y}_{u \cdot x} \underbrace{y}_{u \cdot x} = \underbrace{-\frac{4+1}{4}}_{U \cdot t} \begin{bmatrix} -\frac{4}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

Ex 4: Find the distance from the vector
$$\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$
 to the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ (from Ex 3).

$$\begin{vmatrix} 1 \\ 2 \end{bmatrix} (-\frac{1}{2} + \frac{1}{2} +$$

Notice that the orthogonal projection formula matches the weights of the orthogonal basis terms in theorem 5. Theorem 5 decomposes a vector into a sum of orthogonal projections onto one-dimensional subspaces (lines).

= projection onto u,

 $\hat{\mathbf{y}}_1 = \text{projection onto } \mathbf{u}_1$

In \mathbb{R}^2 , if we have an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ then any $\mathbf{y} \in \mathbb{R}^2$ can be written as

$$\mathbf{y} = rac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + rac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$$

In Physics we use this to decompose force on an object.



A set of vectors $\{\mathbf{u}_{1},...,\mathbf{u}_{p}\}$ is called an <u>orthonormal</u> <u>set</u> if it is an orthogonal set of <u>unit</u> <u>vectors</u>. If W is spanned by this set, then the set is an <u>orthonormal</u> <u>basis</u> for W. Simplest orthonormal basis for \mathbb{R}^{n} is $\{\vec{e}_{1},\vec{e}_{2},...,\vec{e}_{n}\}$. Any nonempty subset of this standard basis is orthonormal as well. **Ex 5:** Determine whether the set of vectors is orthonormal. Is it an orthonormal basis for \mathbb{R}^3 ?

 $U \cdot u = ||\vec{u}||^{2} = \frac{1}{10} + \frac{9}{20} + \frac{9}{20} = 1$ $||v||^{2} = V \cdot V = \frac{9}{10} + \frac{1}{20} + \frac{1}{20} = 1$ $||w||^{2} = W \cdot W = 0 + \frac{1}{2} + \frac{1}{2} = 1$

$$\vec{u} \qquad \vec{v} \qquad \vec{w}$$

$$\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \frac{3}{6} - \frac{3}{20} - \frac{3}{20} = O \checkmark$$

$$\vec{u} \cdot \vec{w} = O - \frac{3}{40} + \frac{3}{40} = O \checkmark$$

$$\vec{v} \cdot \vec{w} = O + \frac{1}{40} - \frac{1}{40} = O \checkmark$$

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$. Proof: $\begin{aligned}
\mathcal{U} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \cdots & \vec{u}_n \cdot \vec{u}_n \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_n \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vec{u}_n \cdot \vec{u}_1 & \cdots & \vec{u}_n \cdot \vec{u}_n \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & - \cdots & 0I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & - \cdots & 0I \end{bmatrix} = \begin{bmatrix} I & n \times n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & - \cdots & 0I \end{bmatrix}$

Theorem 7

Let U be an m imes n matrix with orthonormal columns, and let ${f x}$ and ${f y}$ be in \mathbb{R}^n . Then

- a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$
- b. $(U\mathbf{x})\cdot(U\mathbf{y})=\mathbf{x}\cdot\mathbf{y}$
- c. $(U\mathbf{x})\cdot(U\mathbf{y})=0~$ if and only if $\mathbf{x}\cdot\mathbf{y}=0$

Ex 6: Let
$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
 and $\mathbf{x} = \begin{bmatrix} \sqrt{6} \\ 4 \end{bmatrix}$. Verify that $\|U\mathbf{x}\| \neq \|\mathbf{x}\|$
$$\|U\mathbf{x}\| \neq \|\mathbf{x}\|$$
$$\|U\mathbf{x}\| = \|\mathbf{x}\|$$
$$\|U\mathbf{x}\| = \int 2.2$$
$$\|U\mathbf{x}\| = \int 2.4 + 18 = \int 2.2$$
$$\|V\mathbf{x}\| = \int \frac{\sqrt{2}}{\sqrt{3}} = \begin{bmatrix} \sqrt{2} - \sqrt{3} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} + \sqrt{3} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} + \sqrt{3} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} + \sqrt{3} \end{bmatrix}$$

 $\begin{array}{ll} \text{An} & \underline{Orthogonal} & \underline{Matrix} & \text{is a square invertible matrix } U, \\ \text{such that } U^{-1} = U^T \,. \ \text{By theorem 6, it has orthonormal columns.} & U^\top U = \mathbb{I} \\ \text{The matrix formed from the vectors from Ex 5 is an example.} & \overline{U^\top U} = \mathbb{I} \end{array}$

$$\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 3/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{2} \\ 3/\sqrt{20} & -1/\sqrt{20} & 1/\sqrt{2} \end{bmatrix}$$

Practice Problem

1. Let U and x be as in example 6, and let $\mathbf{y} = \begin{bmatrix} -\sqrt{3} \\ \sqrt{2} \end{bmatrix}$. Verify that $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ $\mathcal{U}_{\mathbf{x}} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ $\mathcal{U}_{\mathbf{y}} = \begin{bmatrix} -1 - i \\ -i \\ -i \\ -i \\ -i \\ +i \end{bmatrix} = \begin{bmatrix} -2 \\ -i \\ 0 \end{bmatrix}$ $(\mathcal{U}_{\mathbf{x}}) \cdot (\mathcal{U}_{\mathbf{y}}) = 2\sqrt{2} - \sqrt{2} + 6 = \boxed{\sqrt{2}}$ $\tilde{\mathbf{x}} \cdot \tilde{\mathbf{y}} = -\sqrt{18} + 4\sqrt{2} = -3\sqrt{2} + 4\sqrt{2} = \boxed{\sqrt{2}}$