6.2 – Orthogonal Sets

Math 220
Warnock - Class Notes

A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called an orthogonal set if each pair of distinct vectors from the set is orthogonal. That is, $u_i \cdot u_j = 0$ when $i \neq j$.

Ex 1: Determine whether the set of vectors is orthogonal.

a) $\overset{u_1}{\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}}$

$u_1 \cdot u_2 = -12 + 21 - 9 = 0 \checkmark$

$u_1 \cdot u_3 = 6 - 7 + 1 = 0 \checkmark$

$\vec{u}_1 \perp \text{Span}\{\vec{u}_2, \vec{u}_3\}$

$u_2 \cdot u_3 = -18 - 3 - 9 = -30 \neq 0$

Not, orthogonal

b) $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

$u_1 \cdot u_2 = -3 - 6 - 3 + 12 = 0 \checkmark$

$u_1 \cdot u_3 = 9 - 16 + 7 + 0 = 0 \checkmark$

$u_2 \cdot u_3 = -3 + 24 - 21 + 0 = 0 \checkmark$

Orthogonal set

Theorem 4
If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

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Proof: Let $\vec{0} = c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p$ (Show $c_i = 0, 1 \leq i \leq p$)

$\vec{0} \cdot \vec{u}_1 = (c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p) \cdot \vec{u}_1$

$\vec{0} = c_1\vec{u}_1 \cdot \vec{u}_1 + \underbrace{c_2\vec{u}_2 \cdot \vec{u}_1 + \ldots + c_p\vec{u}_p \cdot \vec{u}_1}_{\vec{u}_i \cdot \vec{u}_1 = 0,\ i \neq 1}$

$\vec{0} = c_1\vec{u}_1 \cdot \vec{u}_1$ ($\vec{u}_1 \cdot \vec{u}_1 \neq 0$, since $\vec{u}_1$ is non-zero vector)

$\Rightarrow c_1 = 0$

Similarly, $c_2 = c_3 = \ldots = c_p$ are zero $\therefore$ S is linearly independent

Definition
An **orthogonal basis** for a subspace *W* of $\mathbb{R}^n$ is a basis for *W* that is also an orthogonal set.

Theorem 5
Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace *W* of $\mathbb{R}^n$. For each **y** in *W*, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

$y \cdot u_j = c_j u_j \cdot u_j$

$u_i \cdot u_j = 0 \quad \forall i \neq j$

are given by

$$\boxed{c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}} \quad (j = 1, \ldots, p)$$

Ex 2: The vector $\mathbf{v} = \begin{bmatrix} 4 \\ -8 \\ -10 \\ 17 \end{bmatrix}$ is in the subspace *W* with orthogonal basis from Ex 1b).

$\vec{V} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$

$$\begin{array}{ccc} u_1 & u_2 & u_3 \\ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, & \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, & \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \end{array}$$

Express **v** as a linear combination of the orthogonal basis.

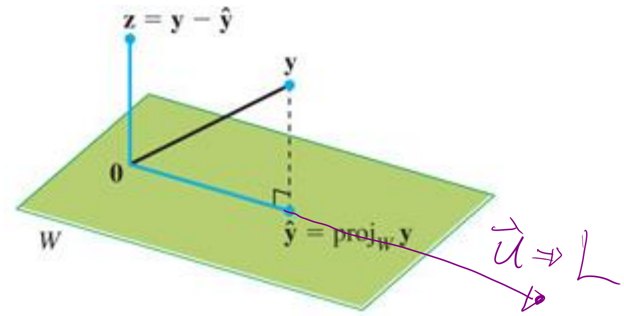
$$c_1 = \frac{\vec{V} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{12+16-10+51}{9+4+1+9} = \frac{69}{23} = 3$$

$$c_2 = \frac{\vec{V} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{-4-24+30+68}{1+9+9+16} = \frac{70}{35} = 2$$

$\vec{V} = 3\vec{u}_1 + 2\vec{u}_2 - \vec{u}_3$

$$c_3 = \frac{\vec{V} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{12-64-70+0}{9+64+49+0} = \frac{-122}{122} = -1$$

An Orthogonal Projection $y = \hat{y} + z$



$\hat{y} = \alpha u$

$z = y - \alpha u$

$0 = z \cdot u = (y - \alpha u) \cdot u$

$0 = y \cdot u - \alpha u \cdot u$

$\alpha = \dfrac{y \cdot u}{u \cdot u} \Rightarrow \boxed{\hat{y} = \dfrac{y \cdot u}{u \cdot u} u}$ ← orthogonal projection of y onto u

$y = \hat{y} + z$ ← component of y orthogonal to u $(z = y - \hat{y})$

$$\boxed{\hat{\mathbf{y}} = \text{proj}_L\, \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}}$$

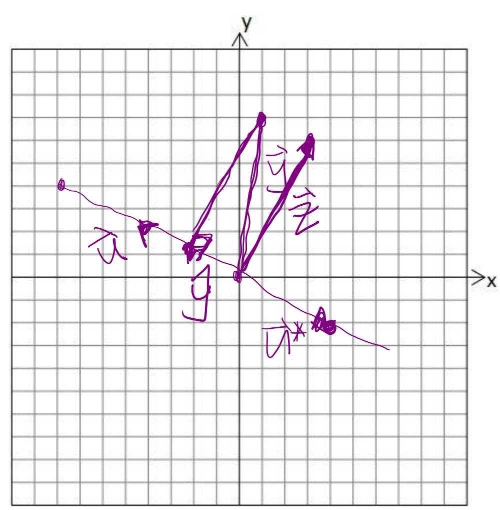
Ex 3: Compute the orthogonal projection of $\vec{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\vec{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

Then write $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ as a sum of two orthogonal vectors. Also, observe geometrically.

$\hat{y} = \dfrac{y \cdot u}{u \cdot u} u = \dfrac{-4+14}{16+4} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \dfrac{1}{2} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$z = y - \hat{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$$\boxed{\begin{gathered} y = \hat{y} + z \\ \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{gathered}}$$



Ex 4: Find the distance from the vector $\begin{bmatrix}1\\7\end{bmatrix}$ to the line through $\begin{bmatrix}-4\\2\end{bmatrix}$ (from Ex 3).

$\hat{y}=\begin{bmatrix}-2\\1\end{bmatrix}$ $\quad \|y-\hat{y}\| = \sqrt{(1-(-2))^2+(7-1)^2} = \sqrt{3^2+6^2} = \sqrt{45} = 3\sqrt{5}$

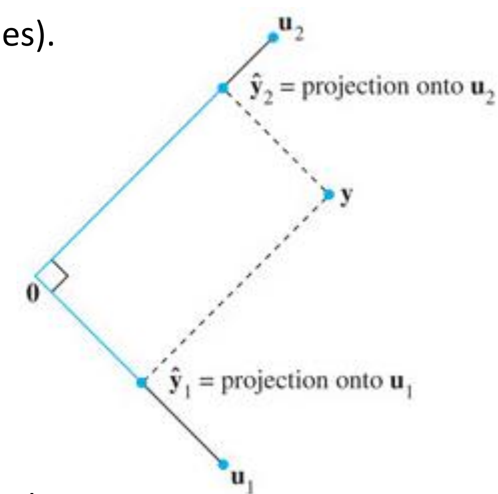
$(=\|z\|)$

Notice that the orthogonal projection formula matches the weights of the orthogonal basis terms in theorem 5. Theorem 5 decomposes a vector into a sum of orthogonal projections onto one-dimensional subspaces (lines).

In $\mathbb{R}^2$, if we have an orthogonal basis $\{\mathbf{u}_1,\mathbf{u}_2\}$

then any $\mathbf{y}\in\mathbb{R}^2$ can be written as

$$\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2$$



In Physics we use this to decompose force on an object.

A set of vectors $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ is called an __orthonormal__ __set__ if it is an orthogonal set of __unit__ __vectors__. If W is spanned by this set, then the set is an __orthonormal__ __basis__ for W.

Simplest orthonormal basis for $\mathbb{R}^n$ is $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$.

Any nonempty subset of this standard basis is orthonormal as well.

Ex 5: Determine whether the set of vectors is orthonormal. Is it an orthonormal basis for $\mathbb{R}^3$?

$$\begin{array}{ccccc} \vec{u} & & \vec{v} & & \vec{w} \\ \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix} & , & \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix} & , & \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \end{array}$$

$u \cdot u = \|\vec{u}\|^2 = \frac{1}{10} + \frac{9}{20} + \frac{9}{20} = 1$ ✓

$\|v\|^2 = v \cdot v = 9/10 + 1/20 + 1/20 = 1$ ✓

$\|w\|^2 = w \cdot w = 0 + \frac{1}{2} + \frac{1}{2} = 1$ ✓

$\vec{u} \cdot \vec{v} = 3/10 - 3/20 - 3/20 = 0$ ✓

$\vec{u} \cdot \vec{w} = 0 - 3/\sqrt{40} + 3/\sqrt{40} = 0$ ✓

$\vec{v} \cdot \vec{w} = 0 + 1/\sqrt{40} - 1/\sqrt{40} = 0$ ✓

Yes, orthor for $\mathbb{R}^3$

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^TU = I$.

Proof:

$U = [\vec{u}_1, \vec{u}_2, \dots \vec{u}_n]$

$$U^TU = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{bmatrix} [\vec{u}_1\ \vec{u}_2, \dots \vec{u}_n] = \begin{bmatrix} \vec{u}_1\cdot\vec{u}_1 & \vec{u}_1\cdot\vec{u}_2 & \cdots & \vec{u}_1\cdot\vec{u}_n \\ \vec{u}_2\cdot\vec{u}_1 & \vec{u}_2\cdot\vec{u}_2 & \cdots & \vec{u}_2\cdot\vec{u}_n \\ \vdots & & \ddots & \vdots \\ \vec{u}_n\cdot\vec{u}_1 & \cdots & \cdots & \vec{u}_n\cdot\vec{u}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = I_{n \times n}$$

$i \neq j \quad \vec{u}_i \cdot \vec{u}_j = 0$

$i = j \quad \vec{u}_i \cdot \vec{u}_j = \vec{u}_i \cdot \vec{u}_i = 1$

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns, and let **x** and **y** be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

Ex 6: Let $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \sqrt{6} \\ 4 \end{bmatrix}$. Verify that $\|U\mathbf{x}\| = \|\mathbf{x}\|$

$$U\vec{x} = \begin{bmatrix} \frac{\sqrt{6}}{\sqrt{3}} - \frac{4}{\sqrt{2}} \\ \frac{\sqrt{6}}{\sqrt{3}} - 0 \\ \frac{\sqrt{6}}{\sqrt{3}} + \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} - 2\sqrt{2} \\ \sqrt{2} \\ \sqrt{2} + 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 3\sqrt{2} \end{bmatrix}$$

$$\|U\vec{x}\| = \sqrt{2+2+18} = \sqrt{22}$$

$$\|\vec{x}\| = \sqrt{6+16} = \sqrt{22}$$

An orthogonal Matrix is a square invertible matrix U, such that $U^{-1} = U^T$. By theorem 6, it has orthonormal columns.

$U^T U = I$

The matrix formed from the vectors from Ex 5 is an example.

$U^{-1} U = I$

$$\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 3/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{2} \\ 3/\sqrt{20} & -1/\sqrt{20} & 1/\sqrt{2} \end{bmatrix}$$

Practice Problem

1. Let U and **x** be as in example 6, and let $\mathbf{y} = \begin{bmatrix} -\sqrt{3} \\ \sqrt{2} \end{bmatrix}$. Verify that $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

$$U\vec{x} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 3\sqrt{2} \end{bmatrix}$$

$$U\vec{y} = \begin{bmatrix} -1-1 \\ -1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$(U\vec{x}) \cdot (U\vec{y}) = 2\sqrt{2} - \sqrt{2} + 0 = \boxed{\sqrt{2}}$$

$$\vec{x} \cdot \vec{y} = -\sqrt{18} + 4\sqrt{2} = -3\sqrt{2} + 4\sqrt{2} = \boxed{\sqrt{2}}$$