6.1 – Inner Product, Length, & Orthogonality

Math 220

Warnock - Class Notes

If **u** and **v** are vectors in $\mathbb{R}^n$ then we can think of them as $n\times 1$ matrices.

So $\mathbf{u}^T$ is a __1xn__ matrix and the product of $\mathbf{u}^T\mathbf{v}$ is a __1 x 1__ matrix.

We will write this as a real number without brackets, and call $\mathbf{u}^T\mathbf{v}$ the __inner product__ of **u** and **v.** It is also written as $\mathbf{u}\cdot\mathbf{v}$ and called the __dot product__.

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Ex 1: Compute $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{u}$ for $\mathbf{u}=\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$

$$\vec{u}^T\vec{v} = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} = 2(-4)+(-3)(2)+4(1) = -8-6+4 = \boxed{-10}$$

$$\vec{v}^T\vec{u} = \begin{bmatrix} -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = -8-6+4 = \boxed{-10}$$

Theorem 1

Let **u**, **v**, and **w** be vectors in $\mathbb{R}^n$, and let *c* be a scalar. Then

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

b. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

c. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

d. $\mathbf{u}\cdot\mathbf{u} \geq 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

$\in \mathbb{R}$ $\qquad$ $\in \mathbb{R}^n$

Generalization of Thm 1

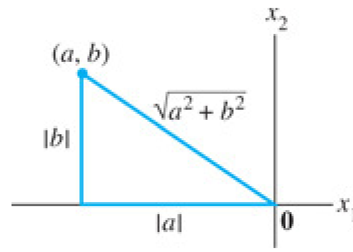
$$\boxed{(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{w} = c_1(\mathbf{u}_1\cdot\mathbf{w}) + \cdots + c_p(\mathbf{u}_p\cdot\mathbf{w})}$$

Definition
The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \text{ and } \boxed{\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}}$$

In $\mathbb{R}^2$ this is essentially the

pythagorean theorem.



$$\boxed{\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|}$$

A vector whose length is one is called the unit vector.

If we divide a non-zero vector **v** by it's length, (multiply by $\frac{1}{\|\vec{v}\|}$) we get a unit vector in the same direction as **v**. This is called normalizing.

$\frac{\vec{v}}{\|\vec{v}\|} = \vec{u}$, unit vector in the same direction as $\vec{v}$.

Ex 2: Let $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ -2 \end{bmatrix}$. Find a unit vector **u** in the same direction as **v.**

$\|\vec{v}\| = \sqrt{5^2 + 2^2 + 4^2 + (-2)^2} = \sqrt{49} = 7$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{7}\begin{bmatrix} 5 \\ 2 \\ 4 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} 5/7 \\ 2/7 \\ 4/7 \\ -2/7 \end{bmatrix}}$$

Ex 3: Let W be a subspace of $\mathbb{R}^2$ spanned by $\mathbf{x} = \begin{bmatrix} 3/4 \\ -2 \end{bmatrix}$. Find a unit vector basis for W.

$\vec{x}^* = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ $\quad \|\vec{x}^*\| = \sqrt{9+64} = \sqrt{73}$

$$\boxed{\vec{u} = \begin{bmatrix} 3/\sqrt{73} \\ -8/\sqrt{73} \end{bmatrix}}$$

or

$\|\vec{x}\| = \sqrt{\frac{9}{16} + 4} = \sqrt{\frac{73}{16}} = \frac{\sqrt{73}}{4}$

$\vec{u} = \frac{1}{\sqrt{73}/4} \begin{bmatrix} 3/4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{73} \\ -8/\sqrt{73} \end{bmatrix}$

How do we find the distance between two numbers on a number line?

-8 0 5

$|5-(-8)|$ or $|-8-5|$

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Definition

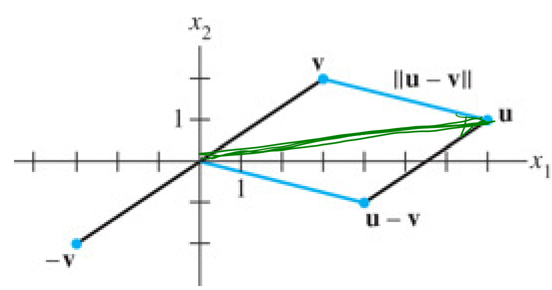
For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, **the distance between u and v**, written as $\operatorname{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Ex 4: Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$.

$\vec{u} - \vec{v} = (4, -1)$

$\|\vec{u} - \vec{v}\| = \|(4,-1)\| = \sqrt{4^2 + (-1)^2} = \boxed{\sqrt{17}}$

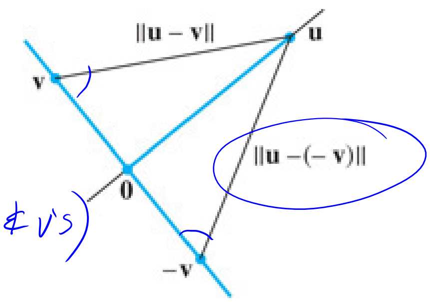


Ex 5: Find the formula for the distance between two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$

$$\|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

(Distance formula in $\mathbb{R}^3$)

Orthogonal Vectors



In $\mathbb{R}^2$ or $\mathbb{R}^3$ two lines through the origin are perpendicular if the distance from **u** to **v** is the same as the distance from **u** to **−v**.

(all bold u & v's)

(square of distances - easier)

$$\|u - (-v)\|^2 = \|u + v\|^2 = (u+v)\cdot(u+v)$$

$$= u\cdot(u+v) + v\cdot(u+v)$$

$$= u\cdot u + u\cdot v + v\cdot u + v\cdot v = \|u\|^2 + \|v\|^2 + 2u\cdot v$$

similarly

$$\|u - v\|^2 = \|u\|^2 + \|-v\|^2 + 2u\cdot(-v)$$

$$= \|u\|^2 + \|v\|^2 - 2u\cdot v$$

$\vec{u}$ orthogonal to $\vec{v}$ only if these are equal

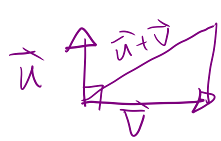
$$\Rightarrow 2u\cdot v = -2u\cdot v$$

$$4u\cdot v = 0$$

$$\vec{u}\cdot\vec{v} = 0$$

Definition
Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u}\cdot\mathbf{v} = 0$.

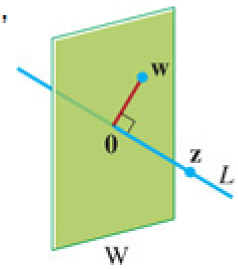
Theorem 2 The Pythagorean Theorem
Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.



If a vector **z** is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then **z** is said to be orthogonal to W. The set of all of these orthogonal vectors to W is called the orthogonal compliment of W and is denoted by $W^\perp$.

Ex 6: Let W be a plane through the origin in $\mathbb{R}^3$, and let L be the line through the origin and perpendicular to W. If **z** and **w** are nonzero, **z** is on L, and **w** is in W, then the line segment from **0** to **z** is perpendicular to the line segment from **0** to **w**; that is, $\mathbf{z} \cdot \mathbf{w} = 0$. See Figure 7. So each vector on L is orthogonal to every **w** in W. In fact, L consists of *all* vectors that are orthogonal to the **w**'s in W, and W consists of all vectors orthogonal to the **z**'s in L. That is,

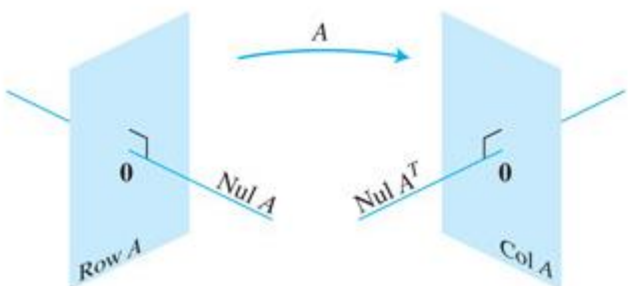
$$L = W^\perp \text{ and } W = L^\perp$$



1. A vector **x** is in $W^\perp$ if and only if **x** is orthogonal to every vector in a set that spans W.

2. $W^\perp$ is a subspace of $\mathbb{R}^n$.

Remember our comment in 4.6 that the Null Space and Row Space are essentially orthogonal to each other.



Theorem 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A \text{ and } (\text{Col } A)^\perp = \text{Nul } A^T$$

Ex 7: Using the Null Space and Row Space of **Ex 5 from 4.6**, check that random vectors from each are orthogonal to each other.

$$\vec{r} = 1\vec{r}_1 + 1\vec{r}_2 + 1\vec{r}_3 = \begin{bmatrix} 1 \\ 1 \\ -7 \\ 1 \\ -3 \end{bmatrix}$$

$$\vec{r}\cdot\vec{n} = -1+9-7+2-3 = 0 \checkmark$$

$$\vec{n} = 1\vec{n}_1 + 1\vec{n}_2 = \begin{bmatrix} -1 \\ 9 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Ex 8: Show that $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\|\cos v$ where v is the angle between the two vectors, using the Law of Cosines,

(bold u's & v's)

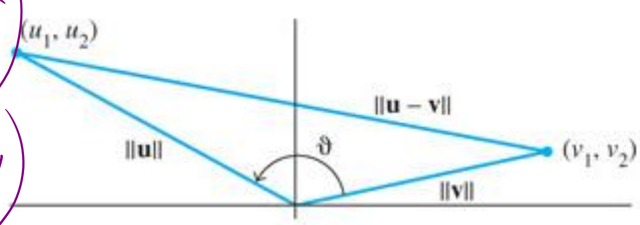
$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\,\underline{\|\mathbf{u}\|\ \|\mathbf{v}\|\cos v}$$

$$\|\vec{u}\|\,\|\vec{v}\|\cos v = -\frac{1}{2}\left(\|u-v\|^2 - \|u\|^2 - \|v\|^2\right)$$

$$= -\frac{1}{2}\left((u-v)\cdot(u-v) - u\cdot u - v\cdot v\right)$$

$$= -\frac{1}{2}\left(u\cdot u - u\cdot v - v\cdot u + v\cdot v - u\cdot u - v\cdot v\right)$$

$$= -\frac{1}{2}\left(-2u\cdot v\right) = \boxed{\vec{u}\cdot\vec{v}}$$



observations

$$\cos v = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

$$v = \pi/2 \Rightarrow \vec{u}\cdot\vec{v} = 0$$