5.3 – Diagonalization

Math 220
Warnock - Class Notes

Ex 1: If $D=\begin{bmatrix}3&0\\0&4\end{bmatrix}$ find D^2, D^3, and D^k.

$$D^2 = D\cdot D = \begin{bmatrix}3&0\\0&4\end{bmatrix}\begin{bmatrix}3&0\\0&4\end{bmatrix} = \begin{bmatrix}3^2&0\\0&4^2\end{bmatrix} = \begin{bmatrix}9&0\\0&16\end{bmatrix}$$

$$D^k = \begin{bmatrix}3^k&0\\0&4^k\end{bmatrix}$$

$$D^3 = D^2\cdot D = \begin{bmatrix}3^2&0\\0&4^2\end{bmatrix}\begin{bmatrix}3&0\\0&4\end{bmatrix} = \begin{bmatrix}3^3&0\\0&4^3\end{bmatrix} = \begin{bmatrix}27&0\\0&64\end{bmatrix}$$

If $A=PDP^{-1}$ for some invertible *P* and diagonal *D*, then A^k is also easy to compute.

Ex 2: Let $A=\begin{bmatrix}7&4\\-3&-1\end{bmatrix}$. Find a formula for A^k given that $A=PDP^{-1}$, where

$\det P = 4$

$P=\begin{bmatrix}-2&-2\\3&1\end{bmatrix}$ and $D=\begin{bmatrix}1&0\\0&5\end{bmatrix}$

$$P^{-1} = \begin{bmatrix}1/4&2/4\\-3/4&-2/4\end{bmatrix}$$

$$A^2 = (PD\underbrace{P^{-1})(P}_{I}DP^{-1})$$

$$= PD^2P^{-1}$$

$$A^3 = (PD^2\underbrace{P^{-1})(P}_{I}DP^{-1})$$

$$= PD^3P^{-1}$$

$\vdots$

$$A^k = PD^kP^{-1}$$

$$PD = \begin{bmatrix}-2&-10\\3&5\end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix}-2&-10\\3&5\end{bmatrix}\begin{bmatrix}1/4&1/2\\-3/4&-1/2\end{bmatrix} = \begin{bmatrix}7&4\\-3&-1\end{bmatrix} \checkmark$$

$$\begin{bmatrix}7&4\\-3&-1\end{bmatrix}^k = \begin{bmatrix}-2&-2\\3&1\end{bmatrix}\begin{bmatrix}1&0\\0&5\end{bmatrix}^k\begin{bmatrix}1/4&1/2\\-3/4&-1/2\end{bmatrix}$$

A square matrix is said to be <u>diagonalizable</u> if A is similar to a diagonal matrix.

Theorem 5 The Diagonalization Theorem
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

These eigenvectors, since they are linearly independent, form a <u>basis of $\mathbb{R}^n$</u>.

Ex 3: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. That is, find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. The eigenvalues are $\lambda = 1, 5$.

1, 1, 5

$\lambda = 1$

$(A - I)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -2 & 1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -2x_2 + x_3$

$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1$ has mult 2

$AP = PD$

$\lambda = 5$

$(A - 5I)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -x_3$
$x_2 = -x_3$

$\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$$\boxed{P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$

$P = \begin{bmatrix} \vec{x}_{\lambda_1}, \vec{x}_{\lambda_2}, \vec{x}_{\lambda_3} \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$0 = (\lambda-4)^2(\lambda-5)$$

Ex 4: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

$\lambda = 4$ (mult 2), 5

$\lambda = 4$

$A - 4I \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$x_1 = 0$

$x_3 = 0$

x_2 is arbitrary

$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 4$, eigenspace has dim 1

mult of 2

Not Diagonalizable

(only have 2 eigenvectors, we need 3)

Theorem 6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Not a requirement though for diagonalizable though, as we saw in Ex 3.

Theorem 7

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

$p \leq n$

read again

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

Ex 4 dim eigenspace 1

$\lambda = 4$ mult 2

c. If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.

$A = PDP^{-1}$ ← eigenvalues (D), eigenvectors (P)

$\lambda = 5, 3, 2 \,(\text{mult } 2)$

Ex 5: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

$\lambda = 2$

$(A-2I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cccc|c} 3 & -3 & 0 & 9 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -x_3 - x_4$

$x_2 = -x_3 + 2x_4$

$\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$(A-3I) \Rightarrow \left[\begin{array}{cccc|c} 2 & -3 & 0 & 9 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & -3/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = \frac{3}{2} x_2$

$x_3 = 0$

$x_4 = 0$

$\vec{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ or $= x_2 \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

$(A-5I) \Rightarrow \left[\begin{array}{cccc|c} 0 & -3 & 0 & 9 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\boxed{P = \begin{bmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}}$$

Check

$AP = PD$

$AP = PD$

Practice Problems

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of *A*. Use this information to diagonalize *A*.

3. Let *A* be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if *A* is diagonalizable?

1

$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

$(4-\lambda)(-1-\lambda) - (-6) = 0$

$-4 - 3\lambda + \lambda^2 + 6 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-1)(\lambda-2) = 0$

$\lambda = 1$

$\begin{bmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = x_2$

$\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$

$\begin{bmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = \frac{3}{2} x_2$

$\vec{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = x_2^* \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$

$A^8 = P D^8 P^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 768 \\ 1 & 512 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix} = A^8$

$\boxed{2}$ $A = PDP^{-1}$ $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$ $P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$P^{-1}AP = D$

$D = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\boxed{3}$