5.2 – The Characteristic Equation

Math 220
Warnock - Class Notes

To find eigenvalues of a square matrix, we are finding non-trivial solutions to the equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$. By the invertible matrix theorem, this is the same as finding λ such that $A-\lambda I$ is ___not invertible___. But this occurs when the ___determinant___ is ___zero___.

Ex 1: Find the Eigenvalues of $A=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$.

$$\begin{bmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{bmatrix}\vec{x}=\vec{0}$$

non-trivial solutions

$$\det(A-\lambda I)=0$$

$$0=\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=(5-\lambda)^2-9=25-10\lambda+\lambda^2-9$$

$$=\lambda^2-10\lambda+16$$

$$0=(\lambda-2)(\lambda-8)$$

$$\boxed{\lambda=2,8}$$

Theorem The Invertible Matrix Theorem (continued)
Let *A* be an $n\times n$ matrix. Then *A* is invertible if and only if:

s. The number 0 is *not* an eigenvalue of *A*. ← (notes from 5.1)

t. The determinant of *A* is *not* zero.

Theorem 3 Properties of Determinants

Let *A* and *B* be $n \times n$ matrices.

a. *A* is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If *A* is triangular, then $\det A$ is the product of the entries on the main diagonal of *A*.

e. A row replacement operation on *A* does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

We can now determine when the matrix $A-\lambda I$ is not invertible by solving the characteristic equation, $\det(A-\lambda I)=0$.

Ex 2: Find the Characteristic equation of $A=\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

$$0 = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = (4-\lambda)\left[(3-\lambda)(2-\lambda)-0\right]$$

$$0 = (4-\lambda)(3-\lambda)(2-\lambda)$$

Ex 3: Find the Characteristic equation of $A=\begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -1 & 2 & 3 & 0 \\ 5 & 0 & 1 & -1 \end{bmatrix}$. triangular

$$\begin{vmatrix} 4-\lambda & 0 & 0 & 0 \\ 2 & 3-\lambda & 0 & 0 \\ -1 & 2 & 3-\lambda & 0 \\ 5 & 0 & 1 & -1-\lambda \end{vmatrix} = \boxed{(4-\lambda)(3-\lambda)(3-\lambda)(-1-\lambda)=0}$$

If *A* is an $n\times n$ matrix, then $\det(A-\lambda I)$ is a polynomial of degree n called the characteristic polynomial of *A*.

The eigenvalue of 3 in Ex 3. is said to have multiplicity two because $(3-\lambda)$ occurs twice.

Ex 4: The Characteristic polynomial of a 7×7 matrix is $\lambda^7-8\lambda^5+16\lambda^3$. Find the eigenvalues and their multiplicities.

$$0=\lambda^3(\lambda^4-8\lambda^2+16)$$
$$=\lambda^3(\lambda^2-4)^2$$
$$0=\lambda^3(\lambda-2)^2(\lambda+2)^2$$

eigenvalues

$\lambda=0$, mult 3

$\lambda=2$, mult 2

$\lambda=-2$, mult 2

Similarity

Two $n \times n$ matrices A and B are considered similar if there is an invertible matrix P such that

$$P^{-1}AP = B \quad \text{Or} \quad A = PBP^{-1}$$

We can also write Q for P^{-1} and get

$$QAQ^{-1} = B \quad \text{Or} \quad A = Q^{-1}BQ$$

Changing A into $P^{-1}AP$ is called the similarity transformation.

Theorem 4
If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

$*$

Proof: Let $B = P^{-1}AP$

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

$$\det(B - \lambda I) = \det\left[P^{-1}(A - \lambda I)P\right]$$

$$= \det P^{-1} \cdot \det(A - \lambda I) \cdot \det P$$

$$= \det P^{-1} \det P \det(A - \lambda I) = \det(P^{-1}P)\det(A - \lambda I)$$

$$= \det I \det(A - \lambda I)$$

$$= 1 \det(A - \lambda I)$$

same characteristic polynomial

$\therefore$ same eigenvalues $*$

Warnings:

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

Practice Problem

Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

$$0 = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)+16$$

$$= 2-3\lambda+\lambda^2+16$$

$$= \lambda^2-3\lambda+18$$