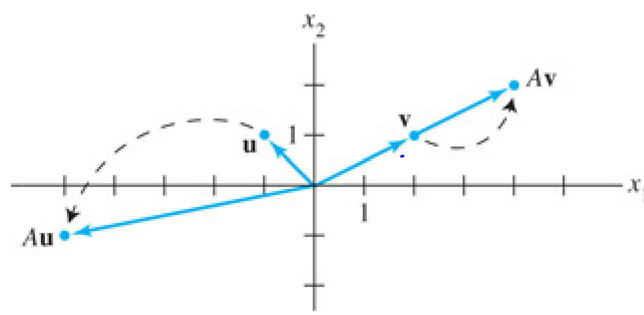
5.1 – Eigenvectors & Eigenvalues

Ex 1: Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Calculate $A\mathbf{u}$ and $A\mathbf{v}$.

What do you notice about either of them?

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A\vec{v} = \lambda \vec{v} = 2\vec{v}$$



Definition
An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

is 3 a solution? $4x = 12$

Ex 2: Is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? If so, find the eigenvalue.

$$\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 30-18 \\ 12-4 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

yes
$\lambda = 4$

Is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? If so, find the eigenvalue.

$$\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 20-9 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix} \neq \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Not an eigenvector

Ex 3: Show that 5 is an eigenvalue of the matrix $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, and find the corresponding eigenvector.

$A\vec{x} = 5\vec{x}$

$A\vec{x} - 5\vec{x} = \vec{0}$

$(A - 5I)\vec{x} = \vec{0}$

$A - 5I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$

$\begin{bmatrix} -4 & 2 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = \frac{1}{2}x_2$

$2x_1 = x_2$

$\vec{x} = c\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$\vec{x} = d\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The eigenvector must be __nonzero__, but an eigenvalue may be __zero__.

So λ is an eigenvalue of an $n \times n$ matrix, if and only if

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

What would another name for the solutions to this equation be?

Nullspace of $A - \lambda I$

But we already know that any __Nullspace__ is a __subspace__ of $\mathbb{R}^n$, so we call it the __eigenspace__ of *A*.

Ex 4: Find a basis for the eigenspace given $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \lambda = 3$

$A\vec{x} = 3\vec{x}$

$A - 3I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

added

$x_1 = -2x_2 - 3x_3$

$\vec{x} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Ex 5: Find the eigenvalues of $\begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$\lambda = 3, 0, 2$

What does it mean for a matrix A to have an eigenvalue of 0?

$A\vec{x} = 0\vec{x} = 0$ has non-trivial solutions

$\Rightarrow$ A is not invertible

This means that 0 is an eigenvalue of A if and only if A is not invertible.

This will be added to our Invertible matrix theorem in 5.2.

Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Proof: Assume $\{\vec{v}_1, \ldots, \vec{v}_r\}$ are linearly dependent

for some $\vec{v}_{p+1} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p$ (first $\vec{v}_{p+1}$ that is a combination) (c's are not all zero)

Multiply by A, and $A\vec{v}_k = \lambda_k \vec{v}_k$, for each k

$c_1 A\vec{v}_1 + \ldots + c_p A\vec{v}_p = A\vec{v}_{p+1}$

$c_1\lambda_1\vec{v}_1 + \ldots c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}$

$*$ by λ_{p+1} and subtract

$-c_1\lambda_{p+1}\vec{v}_1 + \ldots - c_p\lambda_{p+1}\vec{v}_p = -\lambda_{p+1}\vec{v}_{p+1}$

$c_1\underbrace{(\lambda_1 - \lambda_{p+1})}_{\neq 0}\vec{v}_1 + \ldots + c_p\underbrace{(\lambda_p - \lambda_{p+1})}_{\neq 0}\vec{v}_p = \vec{0}$

distinct eigenvalues

since $\vec{v}_1, \ldots, \vec{v}_p$ are lin independent, and $\lambda_k \neq \lambda_{p+1}$ (distinct)

$c_1 = c_2 = \ldots = c_p = 0$

by contradiction, $\{\vec{v}_1, \ldots \vec{v}_r\}$ are linearly independent

Practice Problems

$A\vec{x} = 5\vec{x}$ for some $\vec{x}$

$(A - 5I)\vec{x} = \vec{0}$

has non-trivial solutions

1. Is 5 an eigenvalue of $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?

$$A - 5I = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A - \lambda I$

$\therefore$ not an eigenvalue

(only trivial solution)

(no eigenvector defined)

2. $\underbrace{\text{If } \mathbf{x} \text{ is an eigenvector of } A \text{ corresponding to}}_{\text{means}}$ λ, what is $A^3\mathbf{x}$?

$\lambda^3 \vec{x}$?

$A\vec{x} = \lambda\vec{x}$

$$\begin{aligned} A^3\vec{x} &= A^2(A\vec{x}) \\ &= A^2(\lambda\vec{x}) \\ &= \lambda A^2\vec{x} \\ &= \lambda A(A\vec{x}) \\ &= \lambda A \lambda \vec{x} \\ &= \lambda^2 A\vec{x} \\ &= \lambda^3\vec{x} \end{aligned}$$