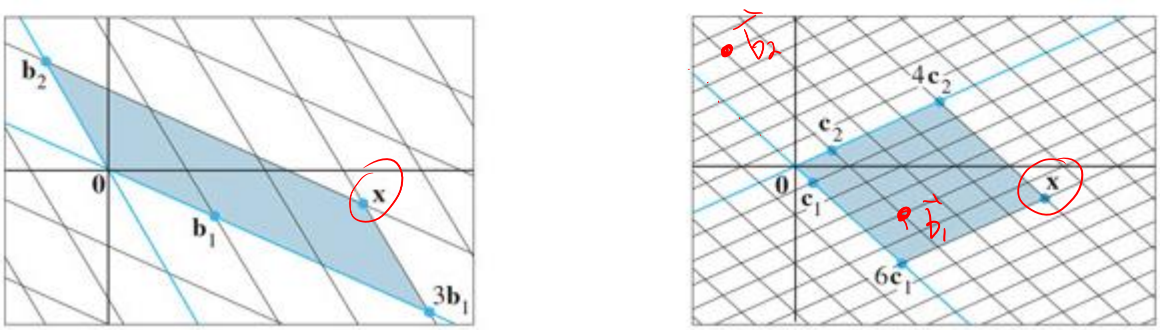
4.7 – Change of Basis

We are now going to look at converting a vector x in one coordinate system into another coordinate system – same vector, different coordinate representation.

Consider the following vector spaces spanned by $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{c}_1, \mathbf{c}_2\}$ respectively.



(a) (b)

By observation, find $[\mathbf{x}]_B = 3\vec{b}_1 + \vec{b}_2$ and $[\mathbf{x}]_C = 6\vec{c}_1 + 4\vec{c}_2$

$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}_B$

Ex 1: Consider two bases $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space *V*, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$ (that is, $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$), find $[\mathbf{x}]_C$.

$$[\vec{x}]_C = [3\vec{b}_1 + \vec{b}_2]_C = 3[\vec{b}_1]_C + [\vec{b}_2]_C$$

$$= \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} = ([x]_C)$$

$$\underset{C \leftarrow B}{P} [\vec{x}]_B = [\vec{x}]_C$$

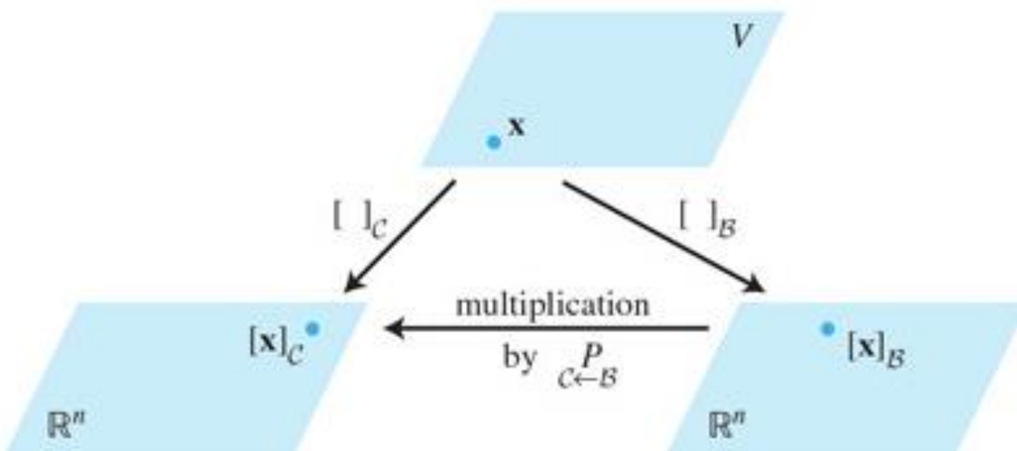
Theorem 15
Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space *V*. Then there is a unique $n \times n$ matrix $\underset{C\leftarrow B}{P}$ such that

$$[\mathbf{x}]_C = \underset{C\leftarrow B}{P}[\mathbf{x}]_B \qquad (4)$$

The columns of $\underset{C\leftarrow B}{P}$ are the *C*-coordinate vectors of the vectors in the basis *B*. That is,

$$\underset{C\leftarrow B}{P} = \begin{bmatrix} [\mathbf{b}_1]_C & [\mathbf{b}_2]_C & \cdots & [\mathbf{b}_n]_C \end{bmatrix} \qquad (5)$$

$\underset{C\leftarrow B}{P}$ is the change of coordinates matrix from B to C



Why are the columns of $\underset{C\leftarrow B}{P}$ linearly independent? B vectors are a basis $\Rightarrow$ linearly independent

So $\underset{C\leftarrow B}{P}$ is Invertible.

So equation (4) above can be re-written as $\left[\underset{C\leftarrow B}{P}\right]^{-1}[\mathbf{x}]_C = [\mathbf{x}]_B$

Since $\underset{C\leftarrow B}{P}$ is the matrix that converts *B*-coordinates to *C*-coordinates, what should $\left[\underset{C\leftarrow B}{P}\right]^{-1}$ do? takes C-coordinates to B-coordinates

$\underset{B\leftarrow C}{P}$

$$\boxed{\left(\underset{C\leftarrow B}{P}\right)^{-1} = \underset{B\leftarrow C}{P}}$$

Change of Basis in $\mathbb{R}^n$

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and ε is the *standard basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_\varepsilon = \mathbf{b}_1$, and likewise for the other vectors in *B*. In this case, $\underset{\varepsilon\leftarrow B}{P}$ is the same as the change-of-coordinates matrix P_B introduced in Section 4.4, namely,

$$P_B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$$

However, to change coordinates between two non-standard bases in $\mathbb{R}^n$, we will need to use Theorem 5, and find coordinate vectors of the old basis relative to the new basis.

Ex 2:

Let $\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from *B* to *C*.

$\left[[\vec{b}_1]_c \; [\vec{b}_2]_c\right] = \underset{C\leftarrow B}{P}$ is c-coordinate vectors of $\vec{b}_1$ & $\vec{b}_2$

$\vec{b}_1 = [\vec{c}_1 \; \vec{c}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\qquad$ $\vec{b}_2 = [\vec{c}_1 \; \vec{c}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$(C^{-1}B)$ also

$\left[\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array}\right] \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{bmatrix} \rightarrow [\vec{b}_1]_c = \begin{bmatrix} 9 \\ -4 \end{bmatrix}, \; [\vec{b}_2]_c = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\boxed{\underset{C\leftarrow B}{P} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}}$$

What does $[\vec{V}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ mean? B-coordinate with $\begin{bmatrix} -6 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\rightarrow \vec{V} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$ $\quad \left(1\begin{bmatrix} -6 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$

$[\vec{V}]_c = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$\vec{V} = 7\begin{bmatrix} 2 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 14-18 \\ -7+6 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

$$\boxed{\left[\begin{array}{cc:cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right] \sim \left[\begin{array}{c:c} I & \underset{C\leftarrow B}{P} \end{array}\right]}$$

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from *C* to *B*. $\underset{B\leftarrow C}{P}$

b. Find the change-of-coordinates matrix from *B* to *C*. $\underset{C\leftarrow B}{P}$

a) $\left[\begin{array}{cc:cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{array}\right]$ $\boxed{\underset{B\leftarrow C}{P} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}}$

or $B^{-1} \Rightarrow \begin{bmatrix} -1/8 & 3/8 \\ -5/8 & 7/8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

Check

$[\vec{v}]_C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\vec{v} = 3\begin{bmatrix} 1 \\ -5 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \end{bmatrix}$

$[\vec{v}]_B = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \end{bmatrix}$

$\vec{v} = -4\begin{bmatrix} 7 \\ 5 \end{bmatrix} - 9\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \end{bmatrix}$ yay!

$\left[\begin{array}{cc:cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \longrightarrow \boxed{\underset{C\leftarrow B}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}}$

$\left(\underset{C\leftarrow B}{P}\right)^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

Practice Problems

1. Let $F = \{\mathbf{f}_1, \mathbf{f}_2\}$ and $G = \{\mathbf{g}_1, \mathbf{g}_2\}$ be bases for a vector space *V*, and let *P* be a matrix whose columns are $[\mathbf{f}_1]_G$ and $[\mathbf{f}_2]_G$. Which of the following equations is satisfied by *P* for all **v** in *V*?

(i) $[\mathbf{v}]_F = P[\mathbf{v}]_G$ $\quad [\vec{v}]_F = \left[\,[\vec{f}_1]_G \; [\vec{f}_2]_G\,\right][\vec{v}]_G$

(ii) $[\mathbf{v}]_G = P[\mathbf{v}]_F$ $\quad [\vec{v}]_G = \left[\,[\vec{f}_1]_G \; [\vec{f}_2]_G\,\right][\vec{v}]_F$

2. Let *B* and *C* be as in Example 1. Use the results of that example to find the change-of-coordinates matrix from *C* to *B*.

$\underset{C\leftarrow B}{P} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \Rightarrow \underset{B\leftarrow C}{P} = \frac{1}{10}\begin{bmatrix} 1 & 6 \\ -1 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1/10 & 3/5 \\ -1/10 & 2/5 \end{bmatrix}}$

Check $\cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$-\frac{6}{10} + \frac{8}{5}$