4.5 – The Dimension of a Vector Space
4.6 – Rank

Math 220

Warnock - Class Notes

Theorem 9

If a vector space V has a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Definition

If V is spanned by a finite set, then V is said to be **finite-dimensional,** and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional.**

Ex 1: Find the following

a) $\dim \mathbb{R}^n = n$

b) $\dim P_3 = \cancel{3}\ 4$ $\qquad \left(P_3 = \text{Span}\{1, t, t^2, t^3\}\right)$

c) $\dim P_n = n+1$

d) $\dim P \cancel{=}$ is infinite dimensional $\qquad$ (P = all polynomials)

e) $\dim H = 2$ $\qquad$ Given $H = \text{span}\left\{\begin{bmatrix}3\\2\\5\end{bmatrix}, \begin{bmatrix}4\\3\\2\end{bmatrix}\right\}$

f) $\dim G = 1$ (lin dep) $\qquad$ Given $G = \text{span}\left\{\begin{bmatrix}5\\-2\\8\end{bmatrix}, \begin{bmatrix}-10\\4\\-16\end{bmatrix}\right\}$

Ex 2: Find the dimension of the subspace

$$a\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}+b\begin{bmatrix}-4\\5\\0\\7\end{bmatrix}+c\begin{bmatrix}-2\\-4\\2\\6\end{bmatrix}$$

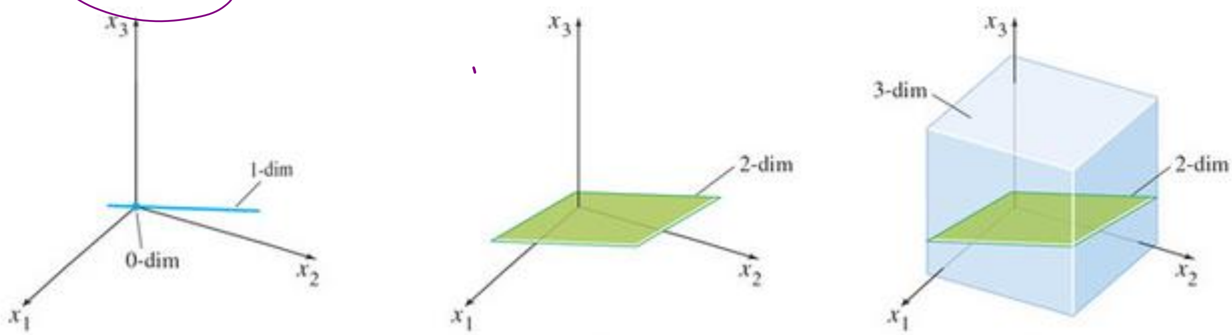
lin dep

$$H=\left\{\begin{bmatrix}a-4b-2c\\2a+5b-4c\\-a+2c\\-3a+7b+6c\end{bmatrix}: a,b,c \text{ in } \mathbb{R}\right\}$$

$\vec{0}\in S$
$\vec{u}+\vec{v}\in S$
$c\vec{u}\in S$

$\dim H = 2 \quad (\vec{b}_3 = -2\vec{b}_1)$

The subspaces of $\mathbb{R}^3$ can be classified by dimension now.



Theorem 11
Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \le \dim V$$

$P \Rightarrow Q$
$\sim Q \Rightarrow \sim P$

$A \Leftrightarrow B$
$\sim A \Leftrightarrow \sim B$

Proof: Case 1: If $H=\{\vec{0}\}$, $\dim H = 0 \le \dim V$

Case 2: Otherwise, let $S=\{\vec{u}_1, \dots \vec{u}_k\}$ be a lin ind set in H.
If it spans H, S is a basis and we're done.
If not, there is some $\vec{u}_{k+1}\in H$ that is not in Span S
so $\{\vec{u}_1, \dots \vec{u}_k, \vec{u}_{k+1}\}$ is lin ind by thm 4
As long as this new set doesn't span H, repeat
Eventually new S will span H, and be a basis.
$\dim H$ won't exceed $\dim V$, otherwise lin dep (Thm 9)

Theorem 12 The Basis Theorem
Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Basis
1. lin ind
2. span set

$\dim V = p$

Proof: ① By Thm 11, any lin ind set of p elements, can be expanded to span V.
since $\dim V = p$, max elements is p, so they span space V

② S has p elements that span V
Spanning Thm says there is a subset $S^* \subset S$ (subset) that is a basis for V
Since $\dim V = p \Rightarrow S^*$ must contain p vectors $\Rightarrow S^* = S$

What can we say about the dimension of Col A and Nul A?

$$\begin{bmatrix} 1 & b & 0 & 0 & c \\ 0 & 0 & 1 & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim \text{Col} A = 3$

$\dim \text{Nul} A = 2$

The dimension of the null space of A is the number of free variables in $A\vec{x} = \vec{0}$

The dimension of the column space of A is: the number of pivot columns in A

Ex 3: Determine the dimensions of the null space and the column space of A.

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim \text{Nul} A = 2$

$\dim \text{Col} A = 3$

Row Space

The set of all the linear combinations of the row vectors of an $m \times n$ matrix *A* is called the row space of *A*, and is denoted by Row *A*. Since there are *n* entries in each row, Row *A* is a subspace of $\mathbb{R}^n$. Also, Row *A* = Col A^T.

Ex 4: Find a spanning set for Row *A*.

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix}$$

$\vec{r}_1 = (1, 0, -3, 1, 2)$

$\vec{r}_2 = (0, 1, -4, -3, 1)$

$\vec{r}_3 = (-3, 2, 1, -8, -6)$

$\vec{r}_4 = (2, -3, 6, 7, 9)$

Row $A = \text{span}\{\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4\}$

Theorem 13
If two matrices *A* and *B* are row equivalent, then their row spaces are the same. If *B* is in echelon form, the nonzero rows of *B* form a basis for the row space of *A* as well as for that of *B*.

Ex 5: Find bases for the row space, column space, and null space of A. $A\vec{x} = \vec{0}$

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 3x_3 - 4x_5$

$x_2 = 4x_3 + 5x_5$

$x_4 = 2x_5$

Row A has basis $R = \{(1,0,-3,0,4), (0,1,-4,0,-5), (0,0,0,1,-2)\}$

Col A has basis $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$

dim Col A = 3

dim Row A = 3

$$\begin{cases} x_1 = 3x_3 - 4x_5 \\ x_2 = 4x_3 + 5x_5 \\ x_4 = 2x_5 \end{cases}$$

Nul A has basis $N = \left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

$\vec{x} = x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

dim Nul A = 2

The rank of A is the dimension of the column space of A.

The rank of A^T is the dimension of the row space of A.

The nullity of A is the dimension of the null space of A (though this text just uses dim Nul A.)

Theorem 14 The Rank Theorem
The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

(See proof on page 235.)

Ex 6: a) If A is an 32 x 10 matrix with three-dimensional null space, what is the rank of A?

rank A = 7

$$\begin{array}{cc} & 10 \\ 32 & \begin{bmatrix} \quad \end{bmatrix} \end{array}$$

b) Could a 3x5 matrix have a one-dimensional null space?

$$\begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Max Rank = 3

Nullity ≥ 2

No!

dim Nul A ≥ (n − m)

In chapter 6 we will learn that Row A and Nul A have only the zero vector in common, and they are actually perpendicular to each other. ***Take a look at example 4 on page 236.***

Ex 7: A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

Does $\underset{(40\times42)(42\times1)}{A\vec{x}} = \vec{b}$ have a solution for all $\vec{b} \in \mathbb{R}^{40}$

nullity A = 2

rank A = 40 ⇒ every $\vec{b} \in \mathbb{R}^{40}$ is spanned by Col A

Theorem The Invertible Matrix Theorem (continued)

Let *A* be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that *A* is an invertible matrix.

m. The columns of *A* form a basis of $\mathbb{R}^n$.

n. $\text{Col } A = \mathbb{R}^n$

o. $\dim \text{Col } A = n$

p. $\text{rank } A = n$

q. $\text{Nul } A = \{\mathbf{0}\}$

r. $\dim \text{Nul } A = 0$

Practice Problems

The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank *A* $=2$ and dim Nul *A* $=3$.

2. Find bases for Col *A* and Row *A*.

$C = \left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$ (or any 2 vectors of A (lin ind) by Basis Thm)

$R = \{(1, -2, -4, 3, -2), (0, 3, 9, -12, 12)\}$

3. What is the next step to perform to find a basis for Nul *A*?

$\frac{1}{3} R_2$

Eliminate -2 in a_{12} with R_2

4. How many pivot columns are in a row echelon form of A^T?

2 pivots

$\text{rank } A^T = \text{rank } A$