4.4 – Coordinate Systems

Math 220
Warnock - Class Notes

*

Theorem 7 The Unique Representation Theorem
Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space *V*. Then for each **x** in *V*, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

Proof: Suppose $\vec{x} = d_1\vec{b}_1 + \dots + d_n\vec{b}_n$ is another representation

$\vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + (c_2 - d_2)\vec{b}_2 + \dots + (c_n - d_n)\vec{b}_n$

Since B is linearly independent (basis)

$c_j - d_j = 0$ for $1 \le j \le n$

$\Rightarrow c_j = d_j$

$\therefore$ $\vec{x}$ has a unique representation.

Definition
Suppose $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for *V* and **x** is in *V*. The **coordinates of x relative to the basis *B*** (or the ***B*-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

We call this vector the coordinate vector of $\vec{x}$ (relative to B)

or the B-coordinate vetor of $\vec{x}$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\boxed{\mathbf{x} \mapsto [\mathbf{x}]_B}$ is the coordinate mapping (determined by *B*)

Ex 1: Consider a basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Suppose an **x** in $\mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find **x**.

$$\vec{x} = -2\vec{b}_1 + 3\vec{b}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Ex 2: The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of **x** relative to the *standard basis* $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, since

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{e}_1 + 6\vec{e}_2$$

If $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, then $[\mathbf{x}]_\varepsilon = \mathbf{x}$.

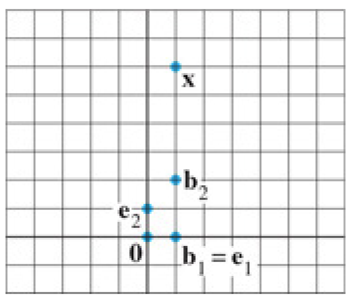


FIGURE 1 Standard graph paper.

$1\vec{e}_1 + 6\vec{e}_2$

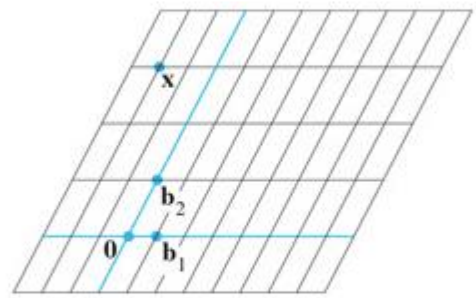


FIGURE 2 *B*-graph paper.

$-2\vec{b}_1 + 3\vec{b}_2$

$-2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$[\vec{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

See Example 3 on page 219.

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_B$ of **x** relative to *B*.

$c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$\underset{\vec{x}}{\begin{bmatrix} 4 \\ 5 \end{bmatrix}} = \underset{P_B}{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}} \underset{[\vec{x}]_B}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

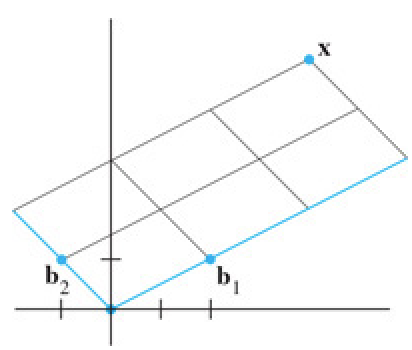
$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 1 & 5 \end{array}\right] \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ $\quad [\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

option 2

$P_B^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$

$P_B^{-1}\vec{x} = [\vec{x}]_B$

$\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 9/3 \\ 6/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = [\vec{x}]_B$



The matrix in (3) changes the *B*-coordinates of a vector **x** into the standard coordinates for **x**. An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Let

$$\boxed{P_B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}}$$

Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$ (unique solution)

is equivalent to

$$\boxed{\mathbf{x} = P_B[\mathbf{x}]_B} \qquad (4)$$

We call P_B the **change-of-coordinates matrix** from *B* to the standard basis in $\mathbb{R}^n$. Left-multiplication by P_B transforms the coordinate vector $[\mathbf{x}]_B$ into **x**.

Since the columns of P_B form a basis, they are linearly independent, and have an inverse, which leads to

$$P_B^{-1}\mathbf{x} = [\mathbf{x}]_B$$ (see previous example again)

The Coordinate Mapping

Choosing a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space *V* introduces a coordinate system in *V*. The coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ connects the possibly unfamiliar space *V* to the familiar space $\mathbb{R}^n$. See Figure 5. Points in *V* can now be identified by their new "names."

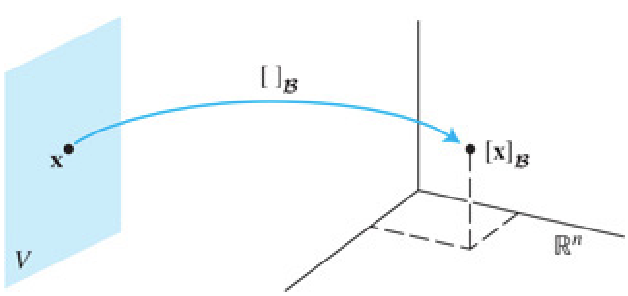


FIGURE 5 The coordinate mapping from *V* onto $\mathbb{R}^n$.

Theorem 8
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space *V*. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from *V* onto $\mathbb{R}^n$.
(see proof in text)

A one-to-one linear transformation from a vector space *V* onto a vector space *W* is called an __isomorphism__ from *V* onto *W*.

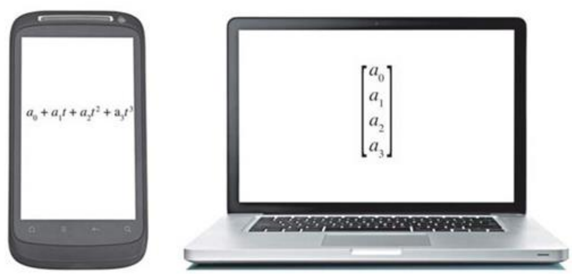
Essentially, these two vector spaces are indistinguishable.

Ex 4: Let *B* be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, let $B = \{1, t, t^2, t^3\}$. A typical element **p** of $\mathbb{P}_3$ has the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since p is a linear combination of the standard basis vectors, then $[\mathbf{p}]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

So $\mathbf{p} \mapsto [\mathbf{p}]_B$ is an isomorphism from $\mathbb{P}_3$ onto $\mathbb{R}^4$.



Ex 5: Use coordinate vectors to test the linear independence of the sets of polynomials.

a) $1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3$

$$[\vec{p}_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad [\vec{p}_2]_B = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix} \quad [\vec{p}_3]_B = \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{lin ind vectors}$$

$\Rightarrow$ lin ind polynomials

Is this a basis for $\mathbb{P}_3$? No, $\mathbb{R}^4$ can't be spanned by 3 vectors
3 poly can't span $\mathbb{P}_3$ (dim 4)

b) $(1-t)^2, t-2t^2+t^3, (1-t)^3$

$1-2t+t^2,\ t-2t^2+t^3,\ 1-3t+3t^2-t^3$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & -3 & 0 \\ 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{lin dependent}$$

$(1-2t+t^2)-(t-2t^2+t^3) = 1-3t+3t^2-t^3$

$P_1 - P_2 = P_3$

$[\vec{P}_1]_B\ [\vec{P}_2]_B\ [\vec{P}_3]_B$

$P_B \vec{x} = \vec{0}$

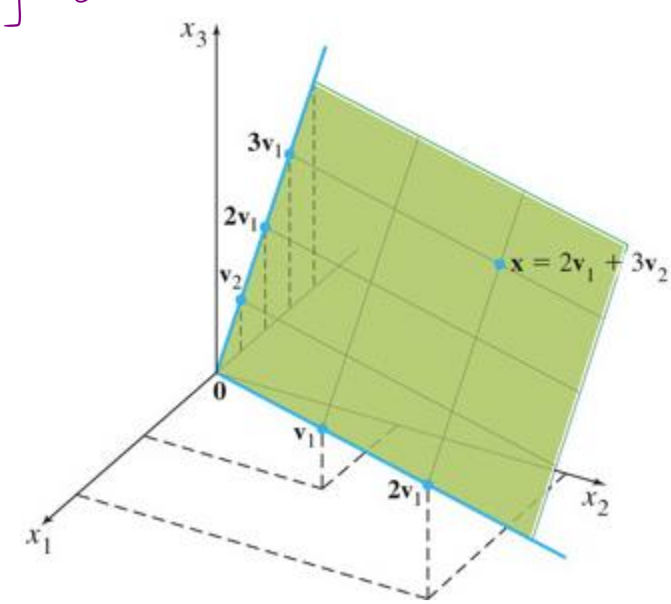
Ex 6: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$,

and $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then *B* is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if **x** is in *H*, and if it is, find the coordinate vector of **x** relative to *B*.

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$

$P_B = \begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad [\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ $\quad \begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix}$

$[\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by observation (or reduce)



Practice Problems

1. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.

a. Show that the set $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$. Yes! 3 pivot positions

b. Find the change-of-coordinates matrix from *B* to the standard basis.

c. Write the equation that relates **x** in $\mathbb{R}^3$ to $[\mathbf{x}]_B$.

d. Find $[\mathbf{x}]_B$, for the **x** given above.

b) $P_B = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

c) $P_B [\vec{x}]_B = \vec{x}$

$$\begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad [\vec{x}]_B = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

$P_B [\vec{p}(t)]_B = \vec{p}(t)$

2. The set $B = \{1+t, 1+t^2, t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6 + 3t - t^2$ relative to *B*.

$$\begin{bmatrix} 1 & 1 & 0 & 6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \underset{=}{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad [\vec{p}(t)]_B = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$$

$$5(1+t) + (1+t^2) - 2(t+t^2) = 6 + 3t - t^2$$