4.3 – Linearly Independent Sets; Bases

Recall the previous definitions of Linearly Independent and Linearly Dependent. We are now going to think in terms of a Vector Space *V*, rather than just $\mathbb{R}^n$.

Definition
An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in ~~$\mathbb{R}^n$~~ V is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

$\vec{0}$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

And recall that

Theorem 4
An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

If a vector space is not just an $\mathbb{R}^n$ with an easy $A\mathbf{x} = \mathbf{0}$, then we need Theorem 4 to show a linear dependence relation to prove linear dependence.

Ex 1: Discuss the linear dependence or independence of the following sets on $C[0,1]$, the space of all continuous functions on $0 \le t \le 1$.

$\{\sin t, \cos t\}$

$$\frac{\sin t}{\cos t} = \frac{a\cos t}{\cos t}$$

$$\tan t = a$$

sin & cos are linearly independent, since there is no dependence relation between them (Thm 4)

$\{\sin t\cos t, \sin 2t\}$

$$\sin 2t = 2\sin t\cos t$$

$\therefore$ lin dep by Thm 4

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) B is a linearly independent set, and

(ii) the subspace spanned by B coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

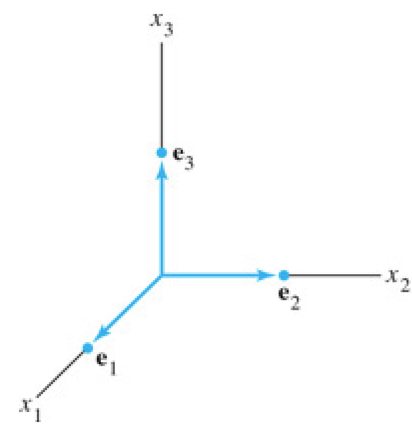
"linearly independent spanning set"

Ex 2: What can we say about an invertible matrix A? (nxn)

1) Columns are linearly independent

2) Columns of A span $\mathbb{R}^n$

Col A = basis for $\mathbb{R}^n$



The columns of the identity matrix, $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$ is called the standard basis for $\mathbb{R}^n$.

Ex 3: Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ 4 & -2 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_3 = \tfrac{1}{2}\vec{v}_1 + 2\vec{v}_2$$

Not a basis

1) Not lin ind

2) Doesn't span $\mathbb{R}^3$

Do $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a basis for $\mathbb{R}^2$?

$\vec{v}_1, \vec{v}_2 \notin \mathbb{R}^2 \Rightarrow$ can't do anything in $\mathbb{R}^2$

$\mathbb{R}^2 \neq \text{span}\{\vec{v}_1, \vec{v}_2\}$

Ex 4: Let $S = \{1, t, t^2, \dots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$. S spans $\mathbb{P}_n$ (all possible terms are present)

$c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n = 0$

$\Leftrightarrow c_0 = c_1 = c_2 = \dots = c_n = 0$

$\Leftrightarrow$ lin ind

$\therefore$ S is a basis for $\mathbb{P}_n$

A basis is an "efficient" spanning set because it contains no unnecessary vectors.

Ex 5: Let $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as in Ex 3. Show that $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \underset{\in}{\overset{\ni}{=}} \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$

($\ni$) $\mathrm{Span}\{\vec{v}_1, \vec{v}_2\} \in H$, since $c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3 \in H$

($\in$) Use $\vec{v}_3 = \frac{1}{2}\vec{v}_1 + 2\vec{v}_2$, (from EX 3)

let $\vec{x} \in H$, (any) $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\left(\frac{1}{2}\vec{v}_1 + 2\vec{v}_2\right)$

$= \left(c_1 + \frac{1}{2}c_3\right)\vec{v}_1 + (c_2 + 2c_3)\vec{v}_2 \in \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$

$\therefore$ $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$

Theorem 5 The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, and let $H = \mathrm{Span}\ \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$ —is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Proof: a) re-arrange to make $\vec{v}_k = \vec{v}_p = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_{p-1}\vec{v}_{p-1}$

let $\vec{x} \in H \Rightarrow \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{p-1}\vec{v}_{p-1} + c_p\vec{v}_p$

$= c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{p-1}\vec{v}_{p-1}$

$c_p a_1\vec{v}_1 + c_p a_2\vec{v}_2 + \dots + c_p a_{p-1}\vec{v}_{p-1}$

$= (c_1 + c_p a_1)\vec{v}_1 + (c_2 + c_p a_2)\vec{v}_2 + \dots + (c_{p-1} + c_p a_{p-1})\vec{v}_{p-1}$

$H = \mathrm{span}\{\vec{v}_1, \dots, \vec{v}_{p-1}\}$

b) H is linearly dependent until it isn't. $\mathrm{span}\{\vec{v}_1, \dots \vec{v}_\ell\} = H$ is no longer linearly dependent $\Rightarrow$ lin ind, spans H

(IF only 1 vector, which isn't $\vec{0}$, therefore lin ind)

$\therefore$ $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_\ell\}$ is a basis

We already know how to find a basis for the Nul A, as we saw that the row reduced system that describes the solutions of Nul A, is already linearly independent.

However, finding a basis for Col A that doesn't have unneeded vectors is our next step.

Ex 6: Find a Basis for Col B where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{b}_3 = -3\vec{b}_1 - 4\vec{b}_2$

$\vec{b}_5 = 4\vec{b}_1 - 5\vec{b}_2 - 2\vec{b}_4$

basis: $H = \text{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$

(pivot columns)

Ex 7: Find a Basis for Col A where, A reduces to the matrix B in the previous example.

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{rref}} B$$

Basis

$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$

Since $A\mathbf{x} = \mathbf{0}$ and the reduced echelon form $B\mathbf{x} = \mathbf{0}$ have the exact same solution sets, then their columns have the exact same dependence relationships. Let's check.

$\vec{b}_3 = -3\vec{b}_1 - 4\vec{b}_2$

$= -3\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}$ ✓

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

WARNING:

You must use the original $\underbrace{\text{pivot columns}}_{\text{Basis, Col A}}$ of A. Why doesn't $\text{Col}A = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$?

No values in 4th position

Theorem 6
The pivot columns of a matrix A form a basis for Col A.

*

↳ linearly independent spanning set

A Basis is basically the smallest spanning set possible. Remove any vectors from it, and the set is no longer spanned, add any vectors to it, and it becomes linearly dependent.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$ — A basis for $\mathbb{R}^3$ — Spans $\mathbb{R}^3$ but is linearly dependent

Practice Problems

1. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-2\\7\\-9\end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

No!

lin ind ✓

span $\mathbb{R}^3$, NO

No, doesn't exist in $\mathbb{R}^2$

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \text{and } \mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace *W* spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

$$\begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \text{Stuff} & \\ 0 & 1 & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis $W = \text{span}\{\vec{v}_1, \vec{v}_2\}$

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{and } H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in *H* is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for *H*? $\boxed{\text{No}}$

lin ind? Yes!

Span set for H? span $\{\vec{v}_1, \vec{v}_2\} = c_1\vec{v}_1 + c_2\vec{v}_2$

$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \text{Span}\{\vec{v}_1, \vec{v}_2\} \notin H$