4.2 – Null Spaces, Column Spaces, and Linear Transformations

Remember that a homogeneous system of equations

$$\begin{aligned} 5x_1 + 21x_2 + 19x_3 &= 0 \\ 13x_1 + 23x_2 + 2x_3 &= 0 \\ 8x_1 + 14x_2 + x_3 &= 0 \end{aligned}$$

Can be written in matrix form as $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}$$

The solution set is all the vectors $\mathbf{x}$ that satisfy the matrix equation. We are going to name this set of solutions the null space.

Definition

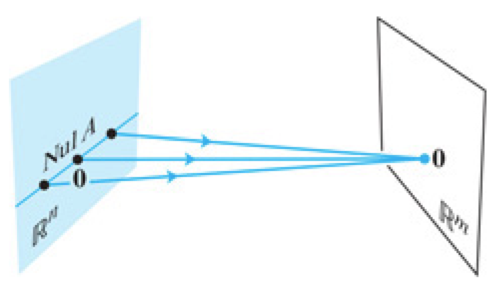
The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Ex 1: Let A be the matrix defined above. Determine whether the vector $\mathbf{u} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ belongs to the null space of A.

$$\begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 - 63 + 38 \\ 65 - 69 + 4 \\ 40 - 42 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yes!



Theorem 2
The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Proof: (1) $\vec{0} \in \text{Nul } A$, yes since $A\vec{0} = \vec{0}$

Let $\vec{u}, \vec{v} \in \text{Nul} A$, $A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$

(2) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0} \Rightarrow \vec{u}+\vec{v} \in \text{Nul} A$

(3) $A(c\vec{u}) = cA\vec{u} = c \cdot \vec{0} = \vec{0} \Rightarrow c\vec{u} \in \text{Nul} A$

$\therefore$ Nul A is a subspace of $\mathbb{R}^n$

Ex 2: Let H be the set of vectors in $\mathbb{R}^3$ whose coordinates a, b, and c satisfy the equations $a - 2b = 5c$ and $2a + 6b = 7c$.
Show that H is a subspace of $\mathbb{R}^3$. *(Create 2 dependence relations between them.)*

$$\begin{aligned} a - 2b - 5c &= 0 \\ 2a + 6b - 7c &= 0 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 1 & -2 & -5 \\ 2 & 6 & -7 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$H = \text{Nul} A$

$\therefore$ H is a subspace of $\mathbb{R}^3$

Ex 3: Find a spanning set for the null space of the matrix $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$. $-3R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

$x_1 = 7x_3 - 6x_4$
$x_2 = -4x_3 + 2x_4$

$\text{Nul} A \in \mathbb{R}^4$

$$\vec{x} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul} A = \text{Span}\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Two properties of null spaces that contain nonzero vectors that we see from the last example.

1. The spanning set is automatically linearly independent
2. The number of vectors in the spanning set of Nul A is equal to the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

Definition
The **column space** of an $m \times n$ matrix *A*, written as Col *A*, is the set of all linear combinations of the columns of *A*. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

Theorem 3
The column space of an $m \times n$ matrix *A* is a subspace of $\mathbb{R}^m$.

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

Ex 4: Find a matrix A such that $W = \text{Col} A$.

$$W = \left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} : b, c, d \text{ real} \right\} \qquad A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Col } A = b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}$$

The column space of an $m \times n$ matrix *A* is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each **b** in $\mathbb{R}^m$.

Ex 5: Given the matrix $A=\begin{bmatrix}1&1&3&1\\2&1&5&4\\1&2&4&-1\end{bmatrix}$, answer the following.

A is $m \times n$ Nul A Col A

a) Find $\mathbb{R}^k$ that Null A is a subspace of.

$\underset{3\times4}{A}\vec{x}=\vec{0} \Rightarrow \vec{x}\in\mathbb{R}^4$ $\boxed{\mathbb{R}^4}$

b) Find $\mathbb{R}^k$ that Col A is a subspace of.

$\mathbb{R}^3$ since collumns of A are in $\mathbb{R}^3$

c) Find a nonzero vector in Null A.

$$\begin{bmatrix}1&1&3&1\\2&1&5&4\\1&2&4&-1\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&2&3\\0&1&1&-2\\0&0&0&0\end{bmatrix}$$

$x_1=-2x_3-3x_4$

$x_2=-x_3+2x_4$

$$\text{Null } A = x_3\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-3\\2\\0\\1\end{bmatrix}$$

Options

$$\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-3\\2\\0\\1\end{bmatrix},\begin{bmatrix}-5\\1\\1\\1\end{bmatrix}$$

d) Find a nonzero vector in Col A.

$$\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}3\\5\\4\end{bmatrix},\begin{bmatrix}5\\8\\7\end{bmatrix}$$

e) Is $\begin{bmatrix}1\\4\\-2\\1\end{bmatrix}$ in the Null A? Is $\begin{bmatrix}2\\0\\2\\5\end{bmatrix}$ in the Null A?

$$\begin{bmatrix}1&1&3&1\\2&1&5&4\\1&2&4&-1\end{bmatrix}\begin{bmatrix}1\\4\\-2\\1\end{bmatrix}=\begin{bmatrix}1+4-6+1\\2+4-10+4\\1+8-8-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\checkmark \text{ yes}$$

No

$$\begin{bmatrix}1&1&3&1\\ & & & \\ & & & \end{bmatrix}\begin{bmatrix}2\\0\\2\\5\end{bmatrix}=\begin{bmatrix}13\\ \cdot\\ \cdot\end{bmatrix}$$

f) Is $\begin{bmatrix}1\\-1\\4\end{bmatrix}$ in Col A?

$$\begin{bmatrix}1&1&3&1&1\\2&1&5&4&-1\\1&2&4&-1&4\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&2&3&-2\\0&1&1&-2&3\\0&0&0&0&0\end{bmatrix}$$

$x_1=-2x_3-3x_4-2$

$x_2=-x_3+2x_4+3$

$x_3=x_4=0$

$x_1=-2$

$x_2=3$

$$-2\begin{bmatrix}1\\2\\1\end{bmatrix}+3\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}1\\-1\\4\end{bmatrix}$$

Contrast Between Nul *A* and Col *A* for an $m \times n$ Matrix *A*

Nul *A*	**Col *A***
1. Nul *A* is a subspace of $\mathbb{R}^n$.	**1.** Col *A* is a subspace of $\mathbb{R}^m$.
2. Nul *A* is implicitly defined; that is, you are given only a condition $(A\mathbf{x} = \mathbf{0})$ that vectors in Nul *A* must satisfy.	**2.** Col *A* is explicitly defined; that is, you are told how to build vectors in Col *A*.
3. It takes time to find vectors in Nul *A*. Row operations on $[A \quad \mathbf{0}]$ are required.	**3.** It is easy to find vectors in Col *A*. The columns of *A* are displayed; others are formed from them.
4. There is no obvious relation between Nul *A* and the entries in *A*.	**4.** There is an obvious relation between Col *A* and the entries in *A*, since each column of *A* is in Col *A*.
5. A typical vector **v** in Nul *A* has the property $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector **v** in Col *A* has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector **v**, it is easy to tell if **v** is in Nul *A*. Just compute *A* **v**.	**6.** Given a specific vector **v**, it may take time to tell if **v** is in Col *A*. Row operations on $[A \quad \mathbf{v}]$ are required.
Square **7.** Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every **b** in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Definition

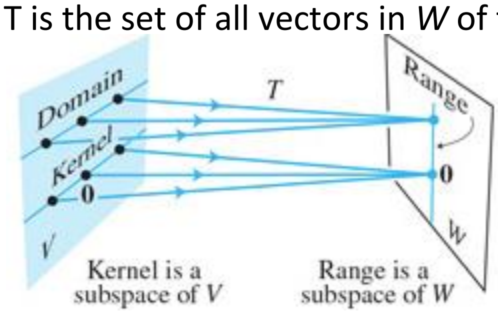
A **linear transformation** *T* from a vector space *V* into a vector space *W* is a rule that assigns to each vector **x** in *V* a unique vector *T* **(x)** in *W*, such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in *V*, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all **u** in *V* and all scalers *c*.

The null space of a linear transformation is called the Kernel, such that $T(\mathbf{u}) = \mathbf{0}$.

The range of T is the set of all vectors in *W* of the form $T(\mathbf{x})$ for some $\mathbf{x} \in V$.



Ex 6:

(Calculus required) Let *V* be the vector space of all real-valued functions *f* defined on an interval $[a,b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a,b]$. Let *W* be the vector space $C[a,b]$ of all continuous functions on $[a,b]$, and let $D: V \to W$ be the transformation that changes *f* in *V* into its derivative f'. In calculus, two simple differentiation rules are

$$D(f+g) = D(f) + D(g) \quad \text{and} \quad D(cf) = cD(f)$$

That is, *D* is a linear transformation. It can be shown that the kernel of *D* is the set of constant functions on $[a,b]$ and the range of *D* is the set *W* of all continuous functions on $[a,b]$.

Practice Problems

1. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show in two different ways that *W* is a subspace of $\mathbb{R}^3$. (Use two theorems.)

$\underbrace{\begin{bmatrix} 1 & -3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ W = Nul A (Like Ex2)

by Thm 2, Nul A is a subspace of $\mathbb{R}^3$

Like Ex4

$a = 3b + c \Rightarrow W = \left\{ b\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, b,c \in \mathbb{R} \right\}$

$W = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow$ By thm 1, spanning set is a subspace of $\mathbb{R}^3$

2. Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose you know that the equations $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = \mathbf{w}$ are both consistent. What can you say about the equation $A\mathbf{x} = \mathbf{v} + \mathbf{w}$?

$\vec{v} \in \text{Col} A$, $\vec{w} \in \text{Col} A$ Col A is a subspace

$\therefore \vec{v} + \vec{w} \in \text{Col} A$

& $A\vec{x} = \vec{v} + \vec{w}$ is consistent