3.1 & 3.2 – Determinants

Math 220
Warnock - Class Notes

To work with larger determinants, we are first going to define the matrix A_{ij} as the matrix *A* with __row__ __i__ and __column__ __j__ deleted.

Definition
For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of *n* terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of *A*. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Ex 1: Find the determinant $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 0\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix}$

$= 0(\quad) - 4(5-0) + 1(15-(-6))$

$= -20 + 21 = \boxed{1}$

We will now define the $(i, j)-$cofactor of matrix *A* as $C_{ij} = (-1)^{i+j} \det A_{ij}$

So the determinant above can be re-written as

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$C_{32} = (-1)^{3+2} \det A_{32}$

Theorem 1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i th row using the cofactors in (4) is

$$\det\ A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the j th column is

$$\det\ A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Ex 2: Use a cofactor expansion across the third column to compute the determinant.

$$\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 4 \\ 5 & -3 \end{vmatrix}$$

$$= 1(15+6) + 1(0-20)$$

$$= 21 - 20 = \boxed{1}$$

Ex 3: Compute the determinant.

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$$

$$3\begin{vmatrix} -2 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} = 3(-2)\begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix} = 3(-2)\big(3(-3) - 4(0)\big)$$

$$= 3(-2)(3)(-3) = \boxed{54}$$

Theorem 2

If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Practice Problem

Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$. $= +2\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$

$= 2(-(-5))\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 10\left(-18-(-20)\right) = 10(2) = \boxed{20}$

3.2

Theorem 3 Row Operations
Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det\ B = \det\ A$.

b. If two rows of A are interchanged to produce B, then $\det\ B = -\det\ A$.

c. If one row of A is multiplied by k to produce B, then $\det\ B = k \cdot \det\ A$.

$\frac{1}{k}\det B = \det A$

Ex 4: Find the determinant by first row-reducing to echelon form.

$\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix} \xrightarrow{\frac{1}{3}R_1} = 3\begin{vmatrix} 1 & 1 & -1 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix} \begin{matrix} -3R_1+R_2 \\ -2R_1+R_3 \end{matrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{vmatrix} 5R_2+R_3$

$= 3\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{vmatrix} = 3\cdot 1\cdot 1\cdot(-8) = \boxed{-24}$

Ex 5: Find the determinant by first row-reducing to echelon form.

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} \begin{matrix} \\ 2R_1+R_2 \\ -3R_1+R_3 \\ -R_1+R_4 \end{matrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} \begin{matrix} \\ \\ \\ -R_3+R_4 \end{matrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$4R_2+R_3 \quad \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 1(1)(30)(0) = \boxed{0}$$

Let's think about a matrix *A* that is row-reduced to echelon form *U* with only row replacements and row interchanges. If we have *r* interchanges, then

$$\det A = (-1)^r \det U$$

Since *U* is in echelon form, it is triangular, so det *U* is just the product of the diagonals.

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \qquad U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\det U \neq 0 \qquad\qquad \det U = 0$$

$$\boxed{\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}}$$

Theorem 4
A square matrix A is invertible if and only if $\det A \neq 0.$

Ex 6: Revisiting Ex 5, at what point could we have stopped?

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} \xrightarrow{\text{1st step}} \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix}$$

Col A are lin dep
$\Rightarrow$ not invertible
$\Rightarrow \det A = 0$

Theorem 5
If A is an $n \times n$ matrix, then $\det A^T = \det A.$

Theorem 6 Multiplicative Property
If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Ex 7: Verify Thm 6 for $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}$

$$AB = \begin{bmatrix} 4-1 & 8+5 \\ 6+4 & 12-20 \end{bmatrix} = \begin{bmatrix} 3 & 13 \\ 10 & -8 \end{bmatrix} \quad \det(AB) = \begin{vmatrix} 3 & 13 \\ 10 & -8 \end{vmatrix} = -24-130 = \boxed{-154} \checkmark$$

$\det A = 2(-4) - 3 = -11$

$\det B = 2(5) - (-1)(4) = 14$

$\Rightarrow * \boxed{-154} \checkmark$

Practice Problems

1. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.

$$\begin{array}{l} -2R_1+R_2 \\ \\ +3R_1+R_4 \end{array} \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix} = 0$$ because $R_2 = R_4$, linearly dependent

2. Use a determinant to decide if $\mathbf{v}_1, \mathbf{v}_2, \text{and } \mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

$$\begin{vmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix} \xrightarrow{R_1+R_2} \begin{vmatrix} -2 & 0 & -5 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix} = {}^{+}-2\begin{vmatrix} 3 & -7 \\ -5 & 5 \end{vmatrix} \overset{+}{-}5\begin{vmatrix} -7 & 3 \\ 9 & -5 \end{vmatrix}$$

$$= -2(15-35) - 5(35-27)$$

$$= 40 - 40 = 0 \Rightarrow \boxed{\text{lin dep}}$$

3. Let *A* be an $n \times n$ matrix such that $A^2 = I$. Show that $\det A = \pm 1$.

$A \cdot A = I$

$\det(A \cdot A) = \det I$

$\det A \cdot \det A = 1$

$(\det A)^2 = 1$

$\det A = \pm\sqrt{1}$

$\det A = \pm 1$ ✓