2.3 – Characteristics of Invertible Matrices

Math 220

Warnock - Class Notes

Theorem 8 The Invertible Matrix Theorem

Let *A* be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given *A*, the statements are either all true or all false.

$A^{-1} \neq \frac{1}{A}$

a. *A* is an invertible matrix.

b. *A* is row equivalent to the $n \times n$ identity matrix.

c. *A* has *n* pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of *A* form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each **b** in $\mathbb{R}^n$.

h. The columns of *A* span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix *C* such that $CA = I$.

k. There is an $n \times n$ matrix *D* such that $AD = I$.

l. A^T is an invertible matrix.

Theorem 5 from 2.2 could also make g. state unique solution.

If *A* and *B* are square matrices, and $AB = I$, then by j. and k. both *A* and *B* are invertible with $B = A^{-1}$ and $A = B^{-1}$.

k & a

$AB = I$, A is invertible

$A^{-1}(AB) = A^{-1}(I)$

$(A^{-1}A)B = A^{-1}$

$IB = A^{-1}$

$B = A^{-1}$

j & a

$AB = I$, B is invertible

$(AB)B^{-1} = (I)B^{-1}$

⋮ (similarly)

$A = B^{-1}$

The Invertible Matrix Theorem essentially divides the set of all $n \times n$ matrices into two disjoint classes:

A has an Inverse	A doesn't have an inverse
Invertible	Not Invertible
nonsingular	singular
n pivot positions	$< n$ pivot positions
Col A lin ind	Col A are lin $\underline{\text{dep}}$
$A\vec{x} = \vec{0}$ has only trivial soln	$A\vec{x} = \vec{0}$ has nontrivial solns
L.T. is onto $\mathbb{R}^n$	L.T. is $\underline{\text{not}}$ onto $\mathbb{R}^n$ (Range $\subset \mathbb{R}^n$, $\mathbb{R}^n \not\subset$ Range)
A^T is invertible	A^T is not invertible
Col A span $\mathbb{R}^n$	Col A don't span $\mathbb{R}^n$
$A\vec{x} = \vec{b}$ has a soln, $\forall \vec{b} \in \mathbb{R}^n$	at least one $\vec{b}$ does not have soln to $A\vec{x} = \vec{b}$
L.T. is 1-to-1	L.T. not one-to-one (multiple inputs for 1 image)
row equivalent to I_n	not row equiv to I_n

Ex 1: Use the Invertible Matrix Theorem to determine if the following are invertible.

$$A = \left[\begin{array}{ccc|ccc} 5 & 2 & 3 & 1 & 0 & 0 \\ 7 & 1 & 2 & 0 & 1 & 0 \\ 11 & -3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{rref}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/19 & 21/76 & -1/76 \\ 0 & 1 & 0 & 5/19 & & \\ 0 & 0 & 1 & & -37/76 & 9/76 \end{array}\right]$$

$$\underbrace{}_{A^{-1} =}$$

3 pivot $\Rightarrow$ invertible

By hand

$$B = \begin{bmatrix} 1 & -3 & -2 \\ 5 & -1 & 18 \\ 4 & 2 & 20 \end{bmatrix} \begin{array}{l} \\ -5R_1 + R_2 \\ -4R_1 + R_3 \end{array}$$

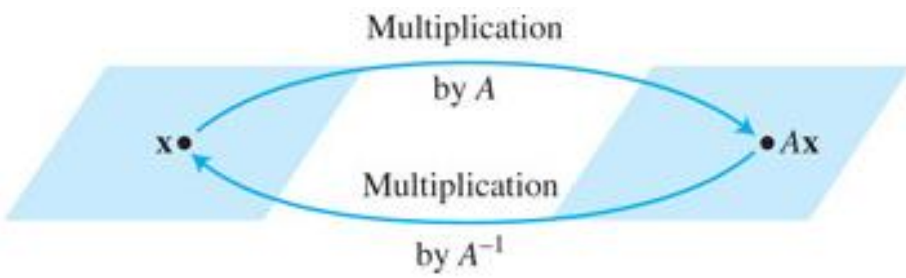
$$= \begin{bmatrix} 1 & -3 & -2 \\ 0 & 14 & 28 \\ 0 & 14 & 28 \end{bmatrix} \begin{array}{l} \\ \\ -R_2 + R_3 \end{array}$$

$$= \begin{bmatrix} 1 & -3 & -2 \\ 0 & 14 & 28 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots, not invertible

Be careful, the Invertible Matrix Theorem only applies to <u>Square</u> matrices.

If A is invertible, we can also think about $A^{-1}A\vec{x} = \vec{x}$ in light of linear transformations.



In general, a Linear Transformation $T:\mathbb{R}^N \to \mathbb{R}^N$ is invertible if there exists a function $S:\mathbb{R}^N \to \mathbb{R}^N$ such that

$$S\big(T(x)\big) = x \quad \text{for all } x \in \mathbb{R}^N$$

$$T\big(S(x)\big) = x \quad \text{for all } x \in \mathbb{R}^N$$

We call *S* the inverse of *T* and write it as T^{-1}.

Theorem 9

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let *A* be the standard matrix for *T*. Then *T* is invertible if and only if *A* is an invertible matrix. In that case, the linear transformation *S* given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

Ex 2: What can be said about a one-to-one linear transformation $T:\mathbb{R}^N \to \mathbb{R}^N$?

T is one-to-one $\iff$ Col A are lin independent

$\iff$ A is invertible

$\iff$ T is invertible

Practice Problems

2. Suppose that for a certain $n \times n$ matrix *A*, statement (g) of the Invertible Matrix Theorem is *not* true. What can you say about equations of the form $A\mathbf{x} = \mathbf{b}$?

At least one $\vec{b}$ has no solution

3. Suppose that *A* and *B* are $n \times n$ matrices and the equation $AB\mathbf{x} = \mathbf{0}$ has a nontrivial solution. What can you say about the matrix *AB*?

AB not invertible

Augmented $A\vec{x} = \vec{b}$

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

General vector form

$x_1 = 3x_2 - 2x_4 + 3$

$x_3 = -x_4 + 2$

$x_5 = -1$

$$\vec{x} = \begin{bmatrix} 3x_2 - 2x_4 + 3 \\ x_2 \\ -x_4 + 2 \\ x_4 \\ -1 \end{bmatrix} = \boxed{x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}}$$